

TWISTED HIGHER INDEX THEORY ON GOOD ORBIFOLDS AND FRACTIONAL QUANTUM NUMBERS

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ABSTRACT. In this paper, we study the twisted higher index theory of elliptic operators on orbifold covering spaces of compact good orbifolds, which are invariant under a projective action of the orbifold fundamental group, and we apply these results to obtain qualitative results, related to generalizations of the Bethe-Sommerfeld conjecture, on the spectrum of self adjoint elliptic operators which are invariant under a projective action of the orbifold fundamental group. We also compute the range of the higher traces on K -theory, which we then apply to compute the range of values of the Hall conductance in the quantum Hall effect on the hyperbolic plane. The new phenomenon that we observe in this case is that the Hall conductance again has plateaus at all energy levels belonging to any gap in the spectrum of the Hamiltonian, where it is now shown to be equal to an integral multiple of a *fractional* valued invariant. Moreover the set of possible denominators is finite and has been explicitly determined. It is plausible that this might shed light on the mathematical mechanism responsible for fractional quantum numbers.

INTRODUCTION

In this paper, we prove a twisted higher index theorem for elliptic operators on orbifold covering spaces of compact good orbifolds, which are invariant under a projective action of the orbifold fundamental group. These are basically the evaluation of pairings of higher traces (which are cyclic cocycles arising from the orbifold fundamental group and the multiplier defining the projective action) with the index of the elliptic operator, considered as an element in the K -theory of some completion of the twisted group algebra of the orbifold fundamental group. The main purpose of generalizing the twisted higher index theorem to orbifolds is to highlight the fact that when the orbifold is not smooth, then the twisted higher index can be a *fraction*. In the smooth case, the higher twisted index theorem was used in [CHMM] to study the quantum Hall effect on hyperbolic space, and one of our key aims in this paper is to generalize the results of [CHMM] to general Fuchsian groups and orbifolds, and can be viewed equivalently as the generalization of results in [CHMM] to the equivariant context. As a result, we obtain a mathematical mechanism that may explain the *fractional quantum numbers* that appear in the quantum Hall effect, cf. section 6.

Let $\Gamma \rightarrow \widetilde{M} \rightarrow M$ be a normal covering space of a compact smooth manifold M . Then in [A] (and clarified by [CM]) Atiyah showed that any Γ -invariant elliptic differential operator \widetilde{P} acting on L^2 sections on \widetilde{M} , and which is the lift of an elliptic differential operator P on M , yields via

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the parametrix construction, an element in K -theory

$$\mathrm{Ind}_\Gamma(\tilde{P}) \in K_0(\mathcal{R}(\Gamma))$$

where $\mathcal{R}(\Gamma)$ is the group ring of Γ with coefficients in the algebra of rapidly decreasing matrices, i.e.

$$\mathcal{R} = \left\{ (a_{ij})_{i,j \in \mathbb{N}} : \sup_{i,j \in \mathbb{N}} i^k j^\ell |a_{ij}| < \infty \ \forall \ k, \ell \in \mathbb{N} \right\}$$

Let tr denote the trace on \mathbb{C} the group algebra $\mathbb{C}(\Gamma)$. More precisely, if δ_g , $g \in \Gamma$ denotes the canonical basis of \mathbb{C} , i.e. $\delta_g(g') = 1$ if $g = g'$ and zero otherwise, then

$$\mathrm{tr}(\delta_g) = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{otherwise} \end{cases}$$

and $\mathrm{Tr} : \mathcal{R} \rightarrow \mathbb{C}$ denote the trace on \mathcal{R} , $\mathrm{Tr}((a_{ij})_{i,j \in \mathbb{N}}) = \sum_{i \in \mathbb{N}} a_{ii}$.

Observe that the natural inclusion map $j : \mathcal{R}(\Gamma) \rightarrow C^*(\Gamma)$ induces a morphism in K -theory

$$j_* : K_\bullet(\mathcal{R}(\Gamma)) \rightarrow K_\bullet(C^*(\Gamma))$$

Then the cup product $\mathrm{tr} \sharp \mathrm{Tr}$ is a trace on $\mathcal{R}(\Gamma)$ which extends to K -theory. We also denote by $\mathrm{tr} \sharp \mathrm{Tr}$ the canonical trace on $C^*(\Gamma)$. Atiyah [At] then proved that

$$[\mathrm{tr} \sharp \mathrm{Tr}](\mathrm{Ind}_\Gamma(\tilde{P})) = [\mathrm{tr} \sharp \mathrm{Tr}](j_*(\mathrm{Ind}_\Gamma(\tilde{P}))) = \mathrm{index}(P)$$

where $\mathrm{index}(P)$ denotes the Fredholm index of the elliptic operator P . Using the Atiyah-Singer index theorem, he was then able to establish a cohomological formula for $[\mathrm{tr} \sharp \mathrm{Tr}](\mathrm{Ind}_\Gamma(\tilde{P}))$.

Since then, there have been two significant generalizations of this theorem. The first is due to Connes and Moscovici [CM], where they compute the pairing of $\mathrm{Ind}_\Gamma(\tilde{P})$ with other higher traces. More precisely, given a normalized group cocycle $c \in Z^k(\Gamma, \mathbb{C})$, they define a cyclic cocycle tr_c on the group ring $\mathbb{C}(\Gamma)$ as follows:

$$\mathrm{tr}_c(\delta_{g_0}, \dots, \delta_{g_k}) = \begin{cases} c(g_1, \dots, g_k) & \text{if } g_0 g_1, \dots, g_k = 1 \\ 0 & \text{otherwise} \end{cases}$$

Then the cup product $\mathrm{tr}_c \sharp \mathrm{Tr}$ extends continuously to a k -dimensional cyclic cocycle on $\mathcal{R}(\Gamma)$ which extends to K -theory, and [CM] establish a cohomological formula for $[\mathrm{tr}_c \sharp \mathrm{Tr}](\mathrm{Ind}_\Gamma(\tilde{P}))$.

Now let σ be a multiplier on Γ and suppose that there is a projective $(\Gamma, \bar{\sigma})$ action on L^2 sections on \widetilde{M} , cf section 1.3. Then in [Gr], Gromov extends Atiyah's index theorem in another direction, to an index theorem for elliptic operators D on \widetilde{M} which are invariant under the projective $(\Gamma, \bar{\sigma})$ action. More precisely, Gromov essentially remarked (and clarified in this paper) that one could modify Atiyah's parametrix construction to obtain a (Γ, σ) -index which is an element in K -theory

$$\mathrm{Ind}_{(\Gamma, \sigma)}(D) \in K_0(\mathcal{R}(\Gamma, \sigma))$$

where $\mathcal{R}(\Gamma, \sigma)$ is the group ring of Γ with coefficients in \mathcal{R} , and which is twisted by the multiplier σ . Let tr denote the trace on $\mathbb{C}(\Gamma, \sigma)$. More precisely, if δ_g , $g \in \Gamma$ denotes the canonical basis of \mathbb{C} , i.e. $\delta_g(g') = 1$ if $g = g'$ and zero otherwise, then

$$\text{tr}(\delta_g) = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{otherwise} \end{cases}.$$

Then the cup product $\text{tr} \# \text{Tr}$ is a trace on $\mathcal{R}(\Gamma, \sigma)$ which extends to K -theory. Observe that the natural inclusion map $j : \mathcal{R}(\Gamma, \sigma) \rightarrow C^*(\Gamma, \sigma)$ induces a morphism in K -theory

$$j_* : K_\bullet(\mathcal{R}(\Gamma, \sigma)) \rightarrow K_\bullet(C^*(\Gamma, \sigma))$$

Gromov also computes a cohomological formula for the (Γ, σ) -index

$$[\text{tr} \# \text{Tr}] (\text{Ind}_{(\Gamma, \sigma)}(D)) = [\text{tr} \# \text{Tr}] (j_*(\text{Ind}_{(\Gamma, \sigma)}(D))).$$

In this paper, we will prove an index theorem which will generalize and unify the index theorems of Atiyah, Connes and Moscovici, and Gromov, in the case of good orbifolds, that is orbifolds such that their orbifold universal cover is a manifold. Now let $\Gamma \rightarrow \widetilde{M} \rightarrow M$ denote the universal orbifold cover of a compact good orbifold M , so that \widetilde{M} is a smooth manifold. Let σ be a multiplier on Γ and assume that there is a projective $(\Gamma, \bar{\sigma})$ -action on L^2 sections of Γ -invariant vector bundles over \widetilde{M} . By considering $(\Gamma, \bar{\sigma})$ -invariant elliptic operators acting on L^2 sections of these bundles, we will show that again, this defines (Γ, σ) -index element in K -theory

$$\text{Ind}_{(\Gamma, \sigma)}(D) \in K_0(\mathcal{R}(\Gamma, \sigma)).$$

We will compute the pairing of $\text{Ind}_{(\Gamma, \sigma)}(D)$ with higher traces. More precisely, given a normalized group cocycle $c \in Z^k(\Gamma, \mathbb{C})$, $k = 0, \dots, \dim M$, we define a cyclic cocycle tr_c of dimension k on the twisted group ring $\mathbb{C}(\Gamma, \sigma)$, which is given by

$$\text{tr}_c(a_0 \delta_{g_0}, \dots, a_k \delta_{g_k}) = \begin{cases} a_0 \dots a_k c(g_1, \dots, g_k) \text{tr}(\delta_{g_0} \delta_{g_1} \dots \delta_{g_k}) & \text{if } g_0 \dots g_k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

where $a_j \in \mathbb{C}$ for $j = 0, 1, \dots, k$. Of particular interest is the case when $k = 2$, when the formula above reduces to

$$\text{tr}_c(a_0 \delta_{g_0}, a_1 \delta_{g_1}, a_2 \delta_{g_2}) = \begin{cases} a_0 a_1 a_2 c(g_1, g_2) \sigma(g_1, g_2) & \text{if } g_0 g_1 g_2 = 1; \\ 0 & \text{otherwise.} \end{cases}$$

The cup product $\text{tr}_c \# \text{Tr}$ extends continuously to a k -dimensional cyclic cocycle on $\mathcal{R}(\Gamma, \sigma)$, which then extends to K -theory. We will also compute a cohomological formula for

$$[\text{tr}_c \# \text{Tr}] (\text{Ind}_{(\Gamma, \sigma)}(D)).$$

Our method consists of applying the Connes-Moscovici *local* higher index theorem to a family of idempotents constructed from the heat operator on \widetilde{M} , all of which represent the (Γ, σ) -index. It is interesting to mention that the orbifold case differs from the smooth case: the index and the L^2 -index are different due to the presence of the singular stratum. The contribution of the singular stratum is present in the index formula, but is not detected by the L^2 -index.

Let Γ be a Fuchsian group of signature (g, ν_1, \dots, ν_n) (cf. section 1 for more details), that is, Γ is the orbifold fundamental group of the 2 dimensional hyperbolic orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$ of signature (g, ν_1, \dots, ν_n) . Using a result of Kasparov [Kas1] on K -amenable groups as well as a calculation by Farsi [Far] of the orbifold K -theory $K_{orb}^\bullet(\Sigma(g, \nu_1, \dots, \nu_n))$ of compact 2-dimensional hyperbolic orbifolds, we are able to compute the K -theory of twisted group C^* algebras, under the assumption that the Dixmier-Douady invariant of the multiplier σ is trivial

$$K_j(C^*(\Gamma, \sigma)) \cong \begin{cases} \mathbb{Z}^{2-n+\sum_{j=1}^n \nu_j} & \text{if } j = 0; \\ \mathbb{Z}^{2g} & \text{if } j = 1. \end{cases}$$

Notice that K_0 is much larger in the general Fuchsian group case than in the torsionfree case, where K_0 was determined to be always \mathbb{Z}^2 , [CHMM]. We also show that the orbifold K -theory of any 2-dimensional orbifold is generated by orbifold line bundles. The result is derived by means of equivariant K -theory and the Baum-Connes [BC] equivariant Chern character with values in the delocalized equivariant cohomology of the smooth surface $\Sigma_{g'}$ that covers the good orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$. We show that the Seifert invariants, cf. [Sc], correspond to the pairing of the equivariant Chern character of [BC] with a fundamental class in the delocalized equivariant homology of $\Sigma_{g'}$.

Using these results and our twisted higher index theorem for orbifolds, we compute in section 3 under the same assumptions as before, the range of the trace on K -theory to be,

$$[\text{tr} \sharp \text{Tr}](K_0(C^*(\Gamma, \sigma))) = \mathbb{Z}\theta + \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i)$$

where θ denotes the evaluation of the multiplier σ on the fundamental class of Γ . We then apply our calculation of the range of the trace on K -theory to study some quantitative aspects of the spectrum of projectively periodic elliptic operators on the hyperbolic plane. Some of the most outstanding open problems about magnetic Schrödinger operators or Hamiltonians on Euclidean space is concerned with the nature of their spectrum, and are the *Bethe-Sommerfeld conjecture* (BSC) and the *Ten Martini Problem* (TMP) (cf. [Sh]). More precisely, TMP asks whether given a multiplier σ on \mathbb{Z}^2 , is there an associated Hamiltonian (*i.e.* a Hamiltonian which commutes with the $(\mathbb{Z}^2, \bar{\sigma})$ projective action of \mathbb{Z}^2 on $L^2(\mathbb{R}^2)$) possessing a Cantor set type spectrum, in the sense that the intersection of the spectrum of the Hamiltonian with some compact interval in \mathbb{R} is a Cantor set? One can deduce from the range of the trace on K_0 of the twisted group C^* -algebras that when the multiplier takes its values in the roots of unity in $U(1)$ (we say then that it is rational) that such a Hamiltonian cannot exist. However, in the Euclidean case and for almost all irrational numbers, the discrete form of TMP has been settled in the affirmative, cf. [Last]. BSC asserts that if the multiplier is trivial, then the spectrum of any associated Hamiltonian has only a *finite* number of gaps. This was first established in the Euclidean case by Skrigonov [Skri]. In Sections 3 and 4, we are concerned also with generalizations of the TMP and the BSC, which we call the *Generalized Ten Dry Martini Problem* and the *Generalized Bethe-Sommerfeld conjecture*. We prove that the Kadison constant of the twisted group C^* -algebra $C_r^*(\Gamma, \sigma)$ is positive whenever the multiplier is rational, where Γ is now the orbifold fundamental group of a signature (g, ν_1, \dots, ν_n) hyperbolic orbifold. We then use the results of Brüning and Sunada [BrSu] to deduce that when the multiplier is rational, the generalized Ten Dry Martini Problem

is answered in the negative, and we leave open the more difficult irrational case. More precisely, we show that the spectrum of such a $(\Gamma, \bar{\sigma})$ projectively periodic elliptic operator is the union of countably many (possibly degenerate) closed intervals which can only accumulate at infinity. This also gives evidence that the generalized Bethe-Sommerfeld conjecture is true, and generalizes earlier results of [CHMM] in the torsion-free case. In section 4, we again use the range of the trace theorem above, together with other geometric arguments to give a complete classification upto isomorphism of the twisted group C^* algebras $C^*(\Gamma, \sigma)$, where σ is assumed to have trivial Dixmier-Douady invariant as before.

In section 5, we use a result of [Ji], which is a twisted analogue of a result of Jollissant and which says in particular that when Γ is a cocompact Fuchsian group, then the natural inclusion map $j : \mathcal{R}(\Gamma, \sigma) \rightarrow C^*(\Gamma, \sigma)$ induces an isomorphism in K -theory

$$K_{\bullet}(\mathcal{R}(\Gamma, \sigma)) \cong K_{\bullet}(C^*(\Gamma, \sigma))$$

Using this, together with our twisted higher index theorem for good orbifolds, and under the same assumptions as before, we are able to compute the range of the higher trace on K -theory

$$[\mathrm{tr}_c \sharp \mathrm{Tr}](K_0(C^*(\Gamma, \sigma))) = \phi \mathbb{Z}$$

where $\phi = 2(g-1) + (n-\nu) \in \mathbb{Q}$, $\nu = \sum_{j=1}^n 1/\nu_j$ and c is the area 2-cocycle on Γ , i.e. c is the restriction to Γ of the area 2-cocycle on $PSL(2, \mathbb{R})$, cf. section 5. We will give examples of good 2-dimensional orbifolds in section 5 for which ϕ is a *fraction*; however it is an *integer* whenever the orbifold is smooth, i.e. whenever $1 = \nu_1 = \dots = \nu_n$, which was the case that was considered in [CHMM].

In section 6, we study the hyperbolic Connes-Kubo formula for the Hall conductance in the discrete model of the Quantum Hall Effect on the hyperbolic plane, where we consider Cayley graphs of Fuchsian groups which may have torsion subgroups, generalizing results in [CHMM] where only torsion-free Fuchsian groups were considered. We recall that the results in [CHMM] generalised to hyperbolic space the noncommutative geometry approach to the Euclidean quantum Hall effect that was pioneered by Bellissard and collaborators [Bel+E+S], Connes [Co] and Xia [Xia]. The Cayley graphs of these Fuchsian groups are not in general trees, as they may now have loops. We first relate the hyperbolic Connes-Kubo cyclic 2-cocycle and the area cyclic 2-cocycle on the algebra $\mathcal{R}(\Gamma, \sigma)$, and show that they define the same class in cyclic cohomology. Then we use the range of the higher trace on K -theory to determine the range of values of the Hall conductance in the Quantum Hall Effect. The new phenomenon that we observe in this case is that the Hall conductance again has plateaus at all energy levels belonging to any gap in the spectrum of the Hamiltonian (known as the generalized Harper operator), where it is now shown to be equal to an integral multiple of a *fractional* valued invariant. Moreover the set of possible denominators is finite and has been explicitly determined, [Bro]. It is plausible that this might shed light on the mathematical mechanism responsible for fractional quantum numbers.

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1. PRELIMINARIES

1.1. Good orbifolds. For further details on the fundamental material on orbifolds, see [Sc], [FuSt] and [Bro].

The definition of an orbifold generalises that of a manifold. More precisely, an *orbifold* M of dimension m is a Hausdorff, second countable topological space with a Satake atlas $\mathcal{V} = \{U_i, \phi_i\}$ which covers M , consisting of open sets U_i and homeomorphisms $\phi_i : U_i \rightarrow D^m/G_i$, where D^m denotes the unit ball in \mathbb{R}^m and G_i is a finite subgroup of the orthogonal group $O(m)$, satisfying the following compatibility relations; the compositions

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \rightarrow \phi_j(U_i \cap U_j)$$

locally lifts to be a smooth map $\mathbb{R}^m \rightarrow \mathbb{R}^m$, whenever the intersection $U_i \cap U_j \neq \emptyset$. The open sets U_i are called local orbifold charts. In general, an orbifold M can be obtained as a quotient $M = X/G$ of an infinitesimally free compact Lie group action on a smooth manifold X . In fact, by Satake [Sat] and Kawasaki [Kaw], X can be chosen to be the smooth manifold of orthonormal frames of the orbifold tangent bundle of M (cf. section 1.4) and G can be chosen to be the orthogonal group $O(m)$.

An orbifold covering of M is an orbifold map $f : Y \rightarrow M$, where Y is also an orbifold, such that any point on M has a neighbourhood U such that $f^{-1}(U)$ is the disjoint union of open sets U_α , with $f|_{U_\alpha} : U_\alpha \rightarrow U$ a quotient map between two quotients of \mathbb{R}^k by finite groups $H_1 < H_2$. The generic fibers of the covering map f are isomorphic to a discrete group which acts as deck transformations.

An orbifold M is *good* if it is orbifold-covered by a smooth manifold; it is *bad* otherwise. A good orbifold is said to be *orientable* if it is orbifold covered by an oriented manifold and the deck transformations act via orientation preserving diffeomorphisms on the orbifold cover. Equivalently, as shown in [Sat] and [Kaw], an orbifold is orientable if it has an oriented frame bundle X such that $M = X/SO(m)$.

We next recall briefly some basic notions on Euclidean and hyperbolic orbifolds, which are by fiat orbifolds whose universal orbifold covering space is Euclidean space and hyperbolic space respectively. We are mainly interested in the case of 2 dimensions, and we will assume that the orbifolds in this paper are orientable.

A 2-dimensional compact orbifold has singularities that are cone points or reflector lines. Up to passing to \mathbb{Z}_2 -orbifold covers, it is always possible to reduce to the case with only isolated cone points.

Let \mathbb{H} denote the hyperbolic plane and Γ a Fuchsian group of signature (g, ν_1, \dots, ν_n) , that is, Γ is a discrete cocompact subgroup of $PSL(2, \mathbb{R})$ of genus g and with n elliptic elements of order ν_1, \dots, ν_n respectively. Explicitly,

$$\Gamma = \left\{ A_i, B_i, C_j \in PSL(2, \mathbb{R}) \mid i = 1, \dots, g, \quad j = 1, \dots, n, \right.$$

$$\left. \prod_{i=1}^g [A_i, B_i] C_1 \dots C_n = 1, \quad C_j^{\nu_j} = 1, \quad j = 1, \dots, n \right\}$$

Then the corresponding compact oriented hyperbolic 2-orbifold of signature (g, ν_1, \dots, ν_n) is defined as the quotient space

$$\Sigma(g, \nu_1, \dots, \nu_n) = \Gamma \backslash \mathbb{H}.$$

A compact oriented 2-dimensional Euclidean orbifold is obtained in a similar manner, but with \mathbb{H} replaced by \mathbb{R}^2 , and a complete list of these can be found in [Sc]. Then $\Sigma(g, \nu_1, \dots, \nu_n)$ is a compact surface of genus g with n elliptic points $\{p_j\}_{j=1}^n$ such that each p_j has a small coordinate neighborhood $U_{p_j} \cong D^2/\mathbb{Z}_{\nu_j}$, where D^2 denotes the unit disk in \mathbb{R}^2 and \mathbb{Z}_{ν_j} is the cyclic group of order ν_j , $j = 1, \dots, n$. Observe that the complement $\Sigma(g, \nu_1, \dots, \nu_n) \setminus \cup_{j=1}^n U_{p_j}$ is a compact Riemann surface of genus g and with n boundary components. The group Γ is the orbifold fundamental group of $\Sigma(g, \nu_1, \dots, \nu_n)$, where the generators C_j can be represented by the n boundary components of the surface $\Sigma(g, \nu_1, \dots, \nu_n) \setminus \cup_{j=1}^n U_{p_j}$.

All Euclidean and hyperbolic 2-dimensional orbifolds $\Sigma(g, \nu_1, \dots, \nu_n)$ are good, being in fact orbifold covered by a smooth surface $\Sigma_{g'}$ cf. [Sc], i.e. there is a finite group G acting on $\Sigma_{g'}$ with quotient $\Sigma(g, \nu_1, \dots, \nu_n)$, where $\#(G) = 1 + \sum_{j=1}^n (\nu_j - 1)$ and $g' = 1 + \frac{\#(G)}{2}(2g - 1) + (n - \nu)$ and where $\nu = \sum_{j=1}^n 1/\nu_j$. According to the classification of 2-dimensional orbifolds given in [Sc], the only bad 2-orbifolds are the “teardrop”, with underlying surface S^2 and one cone point of cone angle $2\pi/p$, and the “double teardrop”, with underlying surface S^2 and two cone points with angles $2\pi/p$ and $2\pi/q$, $p \neq q$.

In this paper we restrict our attention to good orbifolds. It should be pointed out that the techniques used in this paper cannot be extended directly to the case of bad orbifolds. It is reasonable to expect that the index theory on bad orbifolds will involve analytical techniques for more general conic type singularities.

1.2. On Groupoids. Recall that a groupoid consists of a set \mathcal{G} together with a distinguished subset $\mathcal{G}^{(0)} \subset \mathcal{G}$ and two maps

$$r, s : \mathcal{G} \rightarrow \mathcal{G}^{(0)}$$

and a composition law

$$\circ : \mathcal{G}^{(2)} = \{(\gamma_1, \gamma_2) \in \mathcal{G} \times \mathcal{G} : s(\gamma_1) = r(\gamma_2)\} \rightarrow \mathcal{G}$$

such that

1. $s(\gamma_1 \circ \gamma_2) = s(\gamma_2)$ and $r(\gamma_1 \circ \gamma_2) = r(\gamma_1) \forall \gamma_1, \gamma_2 \in \mathcal{G}^{(2)}$
2. $s(x) = r(x) = x \quad \forall x \in \mathcal{G}^{(0)}$
3. $\gamma \circ s(\gamma) = \gamma$ and $r(\gamma) \circ \gamma = \gamma \quad \forall \gamma \in \mathcal{G}$
4. $(\gamma_1 \circ \gamma_2) \circ \gamma_3 = \gamma_1 \circ (\gamma_2 \circ \gamma_3)$
5. Each γ has a 2-sided inverse γ^{-1} such that $\gamma \circ \gamma^{-1} = r(\gamma)$ and $r^{-1} \circ \gamma = s(\gamma)$

r is usually called the *range map* and s the *source map* of the groupoid \mathcal{G} .

1. $\mathcal{G} = \text{group}$, $\mathcal{G}^{(0)} = \{e\}$ and the range and source maps are degenerate;

2. The orbifold fundamental groupoid of M . Let M be a good orbifold, \widetilde{M} be the orbifold universal cover, and Γ be the orbifold fundamental group. We have

$$\begin{aligned}\mathcal{G} &= \widetilde{M} \times_{\Gamma} \widetilde{M}, \quad \mathcal{G}^{(0)} = \widetilde{M}/\Gamma = M \\ r(\tilde{x}, \tilde{y}) &= x \in M, \quad s(\tilde{x}, \tilde{y}) = y \in M \\ (\tilde{x}, \tilde{y}) \circ (\tilde{y}, \tilde{z}) &= (\tilde{x}, \tilde{z}).\end{aligned}$$

Let $u \in \mathcal{G}^{(0)}$ and define $\mathcal{G}^u = r^{-1}(u)$, $\mathcal{G}_u = s^{-1}(u)$

$$\mathcal{G}_u^u = \{\gamma \in \mathcal{G} : u = r(\gamma) = s(\gamma)\} = \mathcal{G}^u \cap \mathcal{G}_u$$

in the example 2. above, $\mathcal{G}^u = \widetilde{M} = \mathcal{G}_u$ and $\mathcal{G}_u^u = \Gamma$.

The orbifold fundamental groupoid of a good orbifold is the main example that we will be concerned with in this paper.

1.3. Twisted Groupoid C^* algebras. Let M be a good, compact orbifold, and $\mathcal{E} \rightarrow M$ be an orbifold vector bundle over M , and $\widetilde{\mathcal{E}} \rightarrow \widetilde{M}$ be its lift to the universal orbifold covering space $\Gamma \rightarrow \widetilde{M} \rightarrow M$, which is by assumption a simply-connected smooth manifold. We will now briefly review how to construct a $(\Gamma, \bar{\sigma})$ -action (where σ is a multiplier on Γ and $\bar{\sigma}$ denotes its complex conjugate) on $L^2(\widetilde{M})$. Let $\omega = d\eta$ be an exact 2-form on \widetilde{M} such that ω is also Γ -invariant, although η is *not* assumed to be Γ -invariant. Define a Hermitian connection on the trivial line bundle over \widetilde{M} as

$$\nabla = d + i\eta$$

Its curvature is $\nabla^2 = i\omega$. Then ∇ defines a $(\Gamma, \bar{\sigma})$ action on $E L^2(\widetilde{M}, \widetilde{\mathcal{S}^{\pm} \otimes E})$ as follows:

Since ω is Γ invariant, one has $\forall \gamma \in \Gamma$

$$0 = \gamma^* \omega - \omega = d(\gamma^* \eta - \eta),$$

so that $\gamma^* \eta - \eta$ is a closed 1-form on a simply-connected manifold $\widetilde{M} \Rightarrow \gamma^* \eta - \eta = d\psi_{\gamma}$ for some smooth function ψ_{γ} on \widetilde{M} satisfying

- $\psi_{\gamma}(x) + \psi_{\gamma'}(\gamma x) - \psi_{\gamma'\gamma}(x)$ is independent of $x \forall x \in \widetilde{M}$
- $\psi_{\gamma}(x_0) = 0$ for some $x_0 \in \widetilde{M} \forall \gamma \in \Gamma$

Then $\bar{\sigma}(\gamma, \gamma') = \exp(i\psi_{\gamma}(\gamma'x_0))$ defines a multiplier on Γ . Now define the $(\Gamma, \bar{\sigma})$ action as follows: For $u \in L^2(\widetilde{M}, \widetilde{\mathcal{S}^{\pm} \otimes E})$, $U_{\gamma} u = \gamma^* u$, $S_{\gamma} u = \exp(i\psi_{\gamma}) u$, define $T_{\gamma} = U_{\gamma} \circ S_{\gamma}$. Then it satisfies $T_{\gamma_1} T_{\gamma_2} = \bar{\sigma}(\gamma_1, \gamma_2) T_{\gamma_1 \gamma_2}$, and so it defines a $(\Gamma, \bar{\sigma})$ -action. It can be shown that only multipliers $\bar{\sigma}$ such that the Dixmier-Douady invariant $\delta(\bar{\sigma}) = 0$ can give rise to $(\Gamma, \bar{\sigma})$ -actions in this way cf. section 3.2 for a further discussion.

Let $D : L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \rightarrow L^2(\widetilde{M}, \widetilde{\mathcal{E}})$ be a self adjoint elliptic differential operator that commutes with a $(\Gamma, \bar{\sigma})$ -action $T_{\gamma} \forall \gamma \in \Gamma$ on $L^2(\widetilde{M}, \widetilde{\mathcal{E}})$. Then by the functional calculus, all the spectral

projections of D , $E_\lambda = \chi_{[0,\lambda]}(D)$ are bounded self adjoint operators on $L^2(\widetilde{M}, \widetilde{\mathcal{E}})$ that commute with $T_\gamma \quad \forall \gamma \in \Gamma$. Now the commutant of the $(\Gamma, \bar{\sigma})$ -action on $L^2(\widetilde{M}, \widetilde{\mathcal{E}})$ is a von Neumann algebra

$$\mathcal{U}(\mathcal{G}, \sigma) = \left\{ Q \in B(L^2(\widetilde{M}, \widetilde{\mathcal{E}})) \mid T_\gamma Q = QT_\gamma \quad \forall \gamma \in \Gamma \right\}$$

where \mathcal{G} denotes the orbifold homotopy groupoid of the orbifold M . Since $T_\gamma Q = QT_\gamma$, one sees that

$$e^{i\phi_\gamma(x)} k_Q(\gamma x, \gamma y) e^{-i\phi_\gamma(y)} = k_Q(x, y)$$

$\forall x, y \in \widetilde{M} \quad \forall \gamma \in \Gamma$, where k_Q denotes the Schwartz kernel of Q . In particular, observe that $\text{tr}(k_Q(x, x))$ is a Γ -invariant function on \widetilde{M} . Using this, one sees that there is a semifinite trace on this von Neumann algebra

$$\text{tr} : \mathcal{U}(\mathcal{G}, \sigma) \rightarrow \mathbb{C}$$

defined as in the untwisted case due to Atiyah [At],

$$Q \rightarrow \int_M \text{tr}(k_Q(x, x)) dx$$

where k_Q denotes the Schwartz kernel of Q . Note that this trace is finite whenever k_Q is continuous in a neighborhood of the diagonal in $\widetilde{M} \times \widetilde{M}$. We remark that $\mathcal{U}(\mathcal{G}, \sigma)$ can also be defined as the weak $*$ -completion of a twisted convolution algebra of functions on the groupoid \mathcal{G} , but we will not have use for this alternate description here.

By elliptic regularity, the spectral projection E_λ has a smooth Schwartz kernel, so that in particular, the spectral density function, $N_Q(\lambda) = \text{tr}(E_\lambda) < \infty \quad \forall \lambda$, is well defined.

If \mathcal{F} is a fundamental domain for the action of Γ on \widetilde{M} , one sees that

$$L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \cong L^2(\Gamma) \otimes L^2(\mathcal{F}, \widetilde{\mathcal{E}}|_{\mathcal{F}})$$

which can be proved by choosing a bounded measurable almost everywhere smooth section of the orbifold covering $\widetilde{M} \rightarrow M$. Therefore it follows easily that

$$\mathcal{U}(\mathcal{G}, \sigma) \cong \mathcal{U}(\Gamma, \sigma) \otimes B(L^2(\mathcal{F}, \widetilde{\mathcal{E}}|_{\mathcal{F}}))$$

where the twisted group von Neumann algebra $\mathcal{U}(\Gamma, \sigma)$ is the weak closure of the twisted group algebra $\mathbb{C}(\Gamma, \sigma)$ and $B(L^2(\mathcal{F}, \widetilde{\mathcal{E}}|_{\mathcal{F}}))$ denotes the algebra of bounded operators on the Hilbert space $L^2(\mathcal{F}, \widetilde{\mathcal{E}}|_{\mathcal{F}})$. There is a natural subalgebra $C^*(\mathcal{G}, \sigma)$ of $\mathcal{U}(\mathcal{G}, \sigma)$ which is defined as follows. Let

$$C_c^\infty(\mathcal{G}, \sigma) = \{ Q \in \mathcal{U}(\mathcal{G}, \sigma) \mid k_Q \text{ is smooth and supported in a compact neighborhood of the diagonal} \}$$

Then $C^*(\mathcal{G}, \sigma)$ is defined to be the *norm closure* of $C_c^\infty(\mathcal{G}, \sigma)$. It can also be shown to be the norm closure of

$$\{ Q \in \mathcal{U}(\mathcal{G}, \sigma) \mid k_Q \text{ is smooth and } k_Q(x, y) \text{ is } L^1 \text{ in both the } x \text{ and } y \text{ variables separately} \}$$

The elements of $C^*(\mathcal{G}, \sigma)$ have the additional property of some off-diagonal decay. In the earlier notation, it can be shown that (cf. [MRW])

$$C^*(\mathcal{G}, \sigma) \cong C^*(\Gamma, \sigma) \otimes \mathcal{K}(L^2(\mathcal{F}, \widetilde{\mathcal{E}}|_{\mathcal{F}}))$$

where the twisted group C^* algebra $C^*(\Gamma, \sigma)$ is the norm closure of the twisted group algebra $\mathbb{C}(\Gamma, \sigma)$ and $\mathcal{K}(L^2(\mathcal{F}, \tilde{\mathcal{E}}|_{\mathcal{F}}))$ denotes the algebra of compact operators on the Hilbert space $L^2(\mathcal{F}, \tilde{\mathcal{E}}|_{\mathcal{F}})$. We remark that $C^*(\mathcal{G}, \sigma)$ can also be defined as the norm completion of a twisted convolution algebra of functions on the groupoid \mathcal{G} , but we will not have use for this alternate description here.

1.4. The C^* algebra of an orbifold. Let M be an oriented orbifold of dimension m , that is $M = P/SO(m)$, where P is the bundle of oriented frames on the orbifold tangent bundle (cf. section 1.4). Then the C^* algebra of the orbifold M is by fiat the crossed product $C^*(M) = C(P) \rtimes SO(m)$, where $C(P)$ denotes the C^* algebra of continuous functions on P . We will now study some Morita equivalent descriptions of $C^*(M)$ that will be useful for us later. The following is one such, and is due to [Far].

Proposition 1.1. *Let M be a good orbifold, which is orbifold covered by the smooth manifold X , i.e. $M = X/G$. Then the C^* algebras $C_0(X) \rtimes G$ and $C^*(M)$ are strongly Morita equivalent.*

In the two dimensional case, there is yet another C^* algebra that is strongly Morita equivalent to the C^* algebra of the orbifold. Let Γ be as before. Then Γ acts freely on $PSL(2, \mathbb{R})$, and therefore the quotient space $\Gamma \backslash PSL(2, \mathbb{R}) = P(g, \nu_1, \dots, \nu_n)$ is a smooth compact manifold, with a right action of $SO(2)$ that is only infinitesimally free. The C^* algebra of the hyperbolic orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$ is by fiat the crossed product C^* algebra

$$C^*(\Sigma(g, \nu_1, \dots, \nu_n)) = C(P(g, \nu_1, \dots, \nu_n)) \rtimes SO(2)$$

cf. [Co]. If $SO(2)$ did act freely on $P(g, \nu_1, \dots, \nu_n)$ (which is the case when $\nu_1 = \dots = \nu_n = 1$), then it is known that $C^*(\Sigma(g, \nu_1, \dots, \nu_n))$ and $C(\Sigma(g, \nu_1, \dots, \nu_n))$ are strongly Morita equivalent as C^* algebras.

We shall next describe a natural algebra which is Morita equivalent to the C^* algebra of the orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$. Now Γ has a torsionfree subgroup $\Gamma_{g'}$ of finite index, such that the quotient $\Gamma_{g'} \backslash \mathbb{H} = \Gamma_{g'} \backslash PSL(2, \mathbb{R}) / SO(2) = \Sigma_{g'}$ is a compact Riemann surface of genus $g' = 1 + \frac{\#(G)}{2}(2(g-1) + (n-\nu))$ where $\#(G) = 1 + \sum_{j=1}^n (\nu_j - 1)$ and where $\nu = \sum_{j=1}^n 1/\nu_j$, cf. Theorem 2.5, [Sc] and the orbifold Euler characteristic calculations in there. Then $G \rightarrow \Sigma_{g'} \rightarrow \Sigma(g, \nu_1, \dots, \nu_n)$ is a finite orbifold cover, i.e. a ramified covering space, where $G = \Gamma_{g'} \backslash \Gamma$.

Proposition 1.2. *The C^* algebras $C(\Sigma_{g'}) \rtimes G$, $C^*(\Sigma(g, \nu_1, \dots, \nu_n))$ and $C_0(\mathbb{H}) \rtimes \Gamma$ are strongly Morita equivalent to each other.*

Proof. The strong Morita equivalence of the last two C^* algebras is contained in the previous Proposition. Since strong Morita equivalence is an equivalence relation, it suffices to prove that the first two C^* algebras are strongly Morita equivalent. Let $\hat{P} = \Gamma_{g'} \backslash PSL(2, \mathbb{R})$ where $SO(2)$ acts on \hat{P} the right, and therefore commutes with the left G action on \hat{P} . Moreover, the actions of G and $SO(2)$ on \hat{P} are free, and therefore one can apply a theorem of Green, [Green] which implies in particular that $C_o(G \backslash \hat{P}) \rtimes SO(2)$ and $C_o(\hat{P}/SO(2)) \rtimes G$ are strongly Morita equivalent, i.e. $C_o(P(g, \nu_1, \dots, \nu_n) \rtimes SO(2))$ and $C_o(\Sigma_{g'}) \rtimes G$ are strongly Morita equivalent, proving the proposition. \square

1.5. Orbifold vector bundles and K -theory. Because of the Morita equivalences of the last section 1.3, we can give several alternate and equivalent descriptions of orbifold vector bundles over orbifolds. Firstly, there is the description using transition functions cf. [Sat], [Kaw]. Equivalently, one can view an orbifold vector bundle over an m dimensional orbifold M as being an $SO(m)$ equivariant vector bundle over the bundle P of oriented frames of the orbifold tangent bundle. In the case of a good orbifold M , which is orbifold covered by a smooth manifold X . Let G be the discrete group acting on X , $G \rightarrow X \rightarrow M = X/G$. Then an orbifold vector bundle on M is the quotient $\mathcal{V}_M = G \backslash \mathcal{V}_X$ of a vector bundle over X by the G action. Notice that an orbifold vector bundle is not a vector bundle over M : in fact, the fibre at a singular point is isomorphic to a quotient of a vector space by a finite group action.

The Grothendieck group of isomorphism classes of orbifold vector bundles on the orbifold M is called the *orbifold K -theory* of M and is denoted by $K_{orb}^0(M)$, which by a result of [BC], [Far] is canonically isomorphic to $K_0(C^*(M))$. By the Morita equivalence of section 1.3, one then has $K_{orb}^0(M) \cong K_{SO(m)}^0(P)$, and by the Julg-Green theorem [Ju], [Green], the second group is isomorphic to $K_0(C(P) \rtimes SO(m))$. In the case when M is a good orbifold, by Proposition 1.1, one sees that $K_{orb}^0(M) \cong K^0(C_0(X) \rtimes G) = K_G^0(X)$.

We will now be mainly interested in orbifold line bundles over the hyperbolic 2-orbifolds. Let G be the finite group determined by the exact sequence $1 \rightarrow \Gamma_{g'} \rightarrow \Gamma \rightarrow G \rightarrow 1$. Then G acts on $\Sigma_{g'}$ with quotient the orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$.

An orbifold line bundle \mathcal{L} on $\Sigma(g, \nu_1, \dots, \nu_n)$ is given by

$$\mathcal{L} = G \backslash (P \times_{SO(2)} \mathbb{C}),$$

where P is a principal $SO(2)$ -bundle on the smooth surface $\Sigma_{g'}$. Notice that the $SO(2)$ and the G actions commute, and are free on the total space P . An orbifold line bundle has an associated Seifert fibred space $G \backslash P$. A more explicit local geometric construction of \mathcal{L} is given in [Sc]. An orbifold line bundle \mathcal{L} over a hyperbolic orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$ is specified by the Chern class of the pullback line bundle on the smooth surface $\Sigma_{g'}$, together with the Seifert data. That is the pairs of numbers (β_j, ν_j) , where β_j satisfies the following condition. Given the exact sequence

$$1 \rightarrow \mathbb{Z} \rightarrow \pi_1(P) \rightarrow \pi_1^{orb}(\Sigma(g, \nu_1, \dots, \nu_n)) \rightarrow 1,$$

let \tilde{C}_j be an element of $\pi_1(P)$ that maps to the generator C_j of the orbifold fundamental group. Let C be the generator of the fundamental group of the fibre. Then we have $C_j^{\nu_j} = 1$ and $C^{\beta_j} = \tilde{C}_j^{\nu_j}$. The choice of β_j can be normalised so that $0 < \beta_j < \nu_j$.

In more geometric terms, let p_1, \dots, p_n be the cone points of a hyperbolic orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$. Let Σ' be the complement of the union of small disks around the cone points. The orbifold line bundle induces a line bundle \mathcal{L}' over the smooth surface with boundary Σ' , trivialized over the boundary components of Σ' . Moreover, the restriction of the orbifold line bundle \mathcal{L} over the small disks D_{p_i} around each cone point p_i is obtained by considering a surgery on the trivial product $\mathbb{C} \times D_{p_i}$ obtained by cutting open along a radius in \mathbb{C} and gluing back after performing a rotation on D_{p_i} by an angle $2\pi q/\nu_i$. With this notation the Seifert invariants are (q_i, ν_i) with $\beta_i q_i \equiv 1 \pmod{\nu_i}$.

Thus, an orbifold line bundle has a finite set of singular fibres at the cone points. The orbifold line bundle \mathcal{L} pulls back to a G -equivariant line bundle $\tilde{\mathcal{L}}$ over the smooth surface $\Sigma_{g'}$ that orbifold covers $\Sigma(g, \nu_1, \dots, \nu_n)$. All the orbifold line bundles with trivial orbifold Euler class, as defined in [Sc], lift to the trivial line bundle on $\Sigma_{g'}$.

In [Sc] the classification of Seifert-fibred spaces is derived using the Seifert invariants, namely the Chern class of the line bundle $\tilde{\mathcal{L}}$, together with the Seifert data (β_j, ν_j) of the singular fibres at the cone points p_j . We show in the following that the Seifert invariants can be recovered from the image of the Baum-Connes equivariant Chern character [BC].

1.6. Baum-Connes Chern character. We have seen that the algebra $C^*(\Sigma(g, \nu_1, \dots, \nu_n))$ is strongly Morita equivalent to the cross product $C(\Sigma_{g'}) \rtimes G$. Therefore the relevant K-theory is

$$K_0(C(\Sigma_{g'}) \rtimes G) = K_{SO(2)}^0(G \backslash \hat{P}) = K_G^0(\Sigma_{g'}),$$

where $\hat{P} = \Gamma_{g'} \backslash PSL(2, \mathbb{R})$.

We recall briefly the definition of delocalised equivariant cohomology for a finite group action on a smooth manifold [BC]. Let G be a finite group acting smoothly and properly on a compact smooth manifold X . Let M be the good orbifold $M = G \backslash X$. Given any $\gamma \in G$, the subset X^γ of X given by

$$X^\gamma = \{(x, \gamma) \in X \times G \mid \gamma x = x\}$$

is a smooth compact submanifold. Let \hat{X} be the disjoint union of the X^γ for $\gamma \in G$. The complex $\Omega_G(\hat{X})$ of G -invariant de Rham forms on \hat{X} with coefficients in \mathbb{C} computes the delocalised equivariant cohomology $H^\bullet(X, G)$, which is \mathbb{Z}_2 graded by forms of even and odd degree. The dual complex that computes delocalised homology is obtained by considering G -invariant de Rham currents on \hat{X} . Thus we have

$$\begin{aligned} H^\bullet(X, G) &= H^\bullet(\Omega_G(\hat{X}), d) = H^\bullet(\hat{X}/G, \mathbb{C}) \\ &= H^\bullet(\hat{X}, \mathbb{C})^G = \bigoplus_{\gamma \in G} H^\bullet(X^\gamma, \mathbb{C}). \end{aligned}$$

According to [BC], Theorem 7.14, the delocalised equivariant cohomology is isomorphic to the cyclic cohomology of the algebra $C^\infty(X) \rtimes G$,

$$\begin{aligned} H^0(X, G) &\cong HC^{ev}(C^\infty(X) \rtimes G), \\ H^1(X, G) &\cong HC^{odd}(C^\infty(X) \rtimes G). \end{aligned}$$

The Baum-Connes equivariant Chern character

$$ch_G : K_G^0(X) \rightarrow H^0(X, G)$$

is an isomorphism over the complex numbers. Equivalently, the Baum-Connes equivariant Chern character can be viewed as

$$ch_G : K_{orb}^0(M) \rightarrow H_{orb}^0(M)$$

where the *orbifold cohomology* is by definition $H_{orb}^j(M) = H^j(X, G)$ for $j = 0, 1$.

In our case the delocalised equivariant cohomology and the Baum-Connes Chern character have a simple expression. Let $\Sigma_{g'}$ be the smooth surface that orbifold covers $\Sigma(g, \nu_1, \dots, \nu_n)$. Let G

be the finite group $1 \rightarrow \Gamma_{g'} \rightarrow \Gamma \rightarrow G \rightarrow 1$. Let $G_{p_j} \cong \mathbb{Z}_{\nu_j}$ be the stabilizer of the cone point p_j in $\Sigma(g, \nu_1, \dots, \nu_n)$. Then we have

$$\Sigma_{g'}^\gamma = \begin{cases} \Sigma_{g'} & \text{if } \gamma = 1; \\ \{p_j\} & \text{if } \gamma \in G_{p_j} \setminus \{1\}; \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus the delocalised equivariant cohomology and orbifold cohomology is given by

$$H_{orb}^0(\Sigma(g, \nu_1, \dots, \nu_n)) = H^0(\Sigma_{g'}, G) = H^0(\Sigma_{g'}) \oplus H^2(\Sigma_{g'}) \oplus \mathbb{C}^{\Sigma_j(\nu_j-1)},$$

where each \mathbb{C}^{ν_j-1} is given by $\nu_j - 1$ copies of $H^0(p_j)$, and

$$H_{orb}^1(\Sigma(g, \nu_1, \dots, \nu_n)) = H^1(\Sigma_{g'}, G) = H^1(\Sigma_{g'}).$$

Let \mathcal{L} be an orbifold line bundle in $K_0(C(\Sigma_g) \rtimes G) = K_G^0(\Sigma_g)$, and let $\tilde{\mathcal{L}}$ be the corresponding line bundle over the surface $\Sigma_{g'}$. An element γ in the stabiliser G_{p_j} acts on the restriction of $\mathcal{L}|_{\Sigma_g^\gamma} = \mathcal{L}|_{p_j} = \mathbb{C}$ as multiplication by $\lambda(\gamma) = e^{2\pi i \beta_j / \nu_j}$.

Thus, the Baum-Connes Chern character of \mathcal{L} is given by

$$ch_G(\mathcal{L}) = (1, c_1(\tilde{\mathcal{L}}), e^{2\pi i \beta_1 / \nu_1}, \dots, e^{2\pi i (\nu_1-1) \beta_1 / \nu_1}, \dots, e^{2\pi i \beta_n / \nu_n}, \dots, e^{2\pi i (\nu_n-1) \beta_n / \nu_n}).$$

Proposition 1.3. *The Baum-Connes Chern character classifies orbifold line bundles over $\Sigma(g, \nu_1, \dots, \nu_n)$.*

Proof. According to [Sc] the orbifold line bundles are classified by the orbifold Euler number

$$e(\Sigma(g, \nu_1, \dots, \nu_n)) = \langle c_1(\tilde{\mathcal{L}}), [\Sigma_{g'}] \rangle + \sum_j \beta_j / \nu_j,$$

given in terms of the Chern number $\langle c_1(\tilde{\mathcal{L}}), [\Sigma_{g'}] \rangle$ and the Seifert invariants (β_j, ν_j) . \square

Notice that we have the isomorphism in K -theory, $K_G^0(\Sigma_{g'}) = K_{SO(2)}^0(G \backslash \hat{P})$ and the Chern character isomorphisms (with \mathbb{C} coefficients)

$$ch_G : K_G^0(\Sigma_{g'}) \rightarrow H^0(\Sigma_{g'}, G) \cong HC^{ev}(C^\infty(\Sigma_{g'}) \rtimes G)$$

and

$$ch_{SO(2)} : K_{SO(2)}^0(\Gamma \backslash PSL(2, \mathbb{R})) \rightarrow HC^{ev}(C^\infty(\Gamma \backslash PSL(2, \mathbb{R})) \rtimes SO(2)).$$

Moreover, we have an isomorphism

$$HC^\bullet(C^\infty(\Gamma \backslash PSL(2, \mathbb{R})) \rtimes SO(2)) \cong H_{SO(2)}^\bullet(\Gamma \backslash PSL(2, \mathbb{R})).$$

Thus, we obtain

$$HC^{ev}(C^\infty(\Gamma \backslash PSL(2, \mathbb{R})) \rtimes SO(2)) \cong HC^{ev}(C^\infty(\Sigma_{g'}) \rtimes G)$$

with \mathbb{C} coefficients, via the Chern character.

Thus orbifold line bundles on $\Sigma(g, \nu_1, \dots, \nu_n)$ can be described equivalently as G -equivariant line bundles over the covering smooth surface $\Sigma_{g'}$, and again as $SO(2)$ -equivariant line bundles on $G \backslash \hat{P}$.

Remarks. With the notation used in the previous section, let G be a finite group acting smoothly and properly on a smooth compact oriented manifold X . There is a natural choice of a fundamental class $[X]_G \in H_0(X, G)$ in the delocalized equivariant homology of X , given by the fundamental classes of each compact oriented smooth submanifold X^γ , $[X]_G = \oplus_{\gamma \in G} [X^\gamma]$. In the case of hyperbolic 2-orbifolds, the equivariant fundamental class $[\Sigma_{g'}]_G$ is given by

$$[\Sigma_{g'}]_G = [\Sigma_{g'}] \oplus_j [p_j]^{\nu_j-1} \in H_2(\Sigma_{g'}, \mathbb{C}) \oplus_j (H_0(p_j, \mathbb{C}))^{\nu_j-1}.$$

The corresponding equivariant Euler number $\langle ch_G(\mathcal{L}), [\Sigma_{g'}]_G \rangle$ is obtained by evaluating

$$\langle ch_G(\mathcal{L}), [\Sigma_{g'}]_G \rangle = \langle c_1(\tilde{\mathcal{L}}), [\Sigma_{g'}] \rangle + \sum_{j=1}^n \sum_{\gamma \in G_{p_j} \setminus \{1\}} \lambda(\gamma).$$

1.7. Classifying space of the orbifold fundamental group. Here we find it convenient to follow Baum, Connes and Higson [BC], [BCH]. Let M be a good orbifold, that is its orbifold universal cover \tilde{M} is a smooth manifold which has a *proper* Γ -action, where Γ denotes the orbifold fundamental group of M . That is, the map

$$\begin{aligned} \tilde{M} \times \Gamma &\rightarrow \tilde{M} \times \tilde{M} \\ (x, \gamma) &\rightarrow (x, \gamma x) \end{aligned}$$

is a proper map. The universal example for such a proper action is denoted in [BC], [BCH] by $\underline{E}\Gamma$. It is universal in the sense that there is a continuous Γ -map

$$f : \tilde{M} \rightarrow \underline{E}\Gamma$$

which is unique upto Γ -homotopy, and moreover $\underline{E}\Gamma$ itself is unique upto Γ -homotopy. The quotient $\underline{B}\Gamma = \Gamma \backslash \underline{E}\Gamma$ is an orbifold. Just as $B\Gamma$ classifies isomorphism classes of Γ -covering spaces, it can be shown that $\underline{B}\Gamma$ classifies isomorphism classes of orbifold Γ -covering spaces.

Examples 1. It turns out that if Γ is a discrete subgroup of a connected Lie group G , then $\underline{E}\Gamma = G/K$, where K is a maximal compact subgroup. This is the main class of examples that we are concerned with in this paper.

Let $S\Gamma$ denote the set of all elements of Γ which are of finite order. Then $S\Gamma$ is not empty, since $1 \in S\Gamma$. Γ acts on $S\Gamma$ by conjugation, and let $F\Gamma$ denote the associated permutation module over \mathbb{C} , i.e.

$$F\Gamma = \left\{ \sum_{\alpha \in S\Gamma} \lambda_\alpha [\alpha] \mid \lambda_\alpha \in \mathbb{C} \text{ and } \lambda_\alpha = 0 \text{ except for a finite number of } \alpha \right\}$$

Let $C^k(\Gamma : F\Gamma)$ denote the space of all antisymmetric $F\Gamma$ -valued Γ -maps on Γ^{k+1} , where Γ acts on Γ^{k+1} via the diagonal action. The coboundary map is

$$\partial c(g_0, \dots, g_{k+1}) = \sum_{i=0}^{k+1} (-1)^i c(g_0, \dots, \hat{g}_i \dots g_{k+1})$$

for all $c \in C^k(\Gamma : F\Gamma)$ and where \hat{g}_i means that g_i is omitted. The cohomology of this complex is the group cohomology of Γ with coefficients in $F\Gamma$, $H^k(\Gamma : F\Gamma)$, cf. [BCH]. They also show that $H^k(\Gamma : F\Gamma) \cong H^j(\Gamma, \mathbb{C}) \oplus_m H^k(Z(C_m) : \mathbb{C})$, where $S\Gamma = \{1, C_m | m = 1, \dots\}$ and the isomorphism is canonical.

Also, for any Borel measurable Γ -map $\mu : \underline{E}\Gamma \rightarrow \Gamma$, there is an induced map on cochains

$$\mu^* : C^k(\Gamma : F\Gamma) \rightarrow C^k(\underline{E}\Gamma : \Gamma)$$

which induces an isomorphism on cohomology, $\mu^* : H^k(\Gamma : F\Gamma) \cong H^k(\underline{E}\Gamma : \Gamma)$ [BCH]. Here $H^j(\underline{E}\Gamma, \Gamma)$ denotes the \mathbb{Z} -graded (delocalised) equivariant cohomology of $\underline{E}\Gamma$, which is a refinement of what was discussed earlier, and which is defined in [BCH] using sheaves (and cosheaves), but we will not recall the definition here.

Let M be a good orbifold with orbifold fundamental group Γ . We have seen that the universal orbifold cover \widetilde{M} is classified by a continuous map $f : M \rightarrow \underline{B}\Gamma$, or equivalently by a Γ -map $f : \widetilde{M} \rightarrow \underline{E}\Gamma$. The induced map is $f^* : H_{orb}^j(\underline{B}\Gamma, \mathbb{C}) \equiv H^k(\underline{E}\Gamma : \Gamma) \rightarrow H^k(\widetilde{M} : \Gamma) \equiv H_{orb}^k(M : \mathbb{C})$ and therefore in particular one has $f^*([c]) \in H_{orb}^k(M : \mathbb{C})$ for all $[c] \in H^k(\Gamma : \mathbb{C})$. This can be expressed on the level of cochains by a easily modifying the procedure in [CM], and we refer to [CM] for further details.

2. TWISTED HIGHER INDEX THEOREM

In this section, we will define the higher twisted index of an elliptic operator on a good orbifold, and establish a cohomological formula for any cyclic trace arising from a group cocycle, and which is applied to the twisted higher index. We adapt the strategy and proof in [CM] to our context.

2.1. Construction of the parametrix and the index map. For basic material on orbifolds, see [Sc] and references there. Let M be a compact, good orbifold, that is, the universal cover $\Gamma \rightarrow \widetilde{M} \rightarrow M$ is a smooth manifold and we will assume as before that there is a $(\Gamma, \bar{\sigma})$ -action on $L^2(\widetilde{M})$ given by $T_\gamma = U_\gamma \circ S_\gamma \forall \gamma \in \Gamma$. Let $\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}$ be Hermitian vector bundles on M and let $\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}$ be the corresponding lifts to Γ -invariants Hermitian vector bundles on \widetilde{M} . Then there are induced (Γ, σ) -actions on $L^2(\widetilde{M}, \widetilde{\mathcal{E}})$ and $L^2(\widetilde{M}, \widetilde{\mathcal{F}})$ which are also given by $T_\gamma = U_\gamma \circ S_\gamma \forall \gamma \in \Gamma$.

Now let $D : L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \rightarrow L^2(\widetilde{M}, \widetilde{\mathcal{F}})$ be a 1st order $(\Gamma, \bar{\sigma})$ -invariant elliptic operator. Let $U \subset \widetilde{M}$ be an open subset that contains the closure of a fundamental domain for the Γ -action on \widetilde{M} . Let $\psi \in C_c^\infty(\widetilde{M})$ be a compactly supported smooth function such that $\text{supp}(\psi) \subset U$, and

$$\sum_{\gamma \in \Gamma} \gamma^* \psi = 1.$$

Let $\phi \in C_c^\infty(\widetilde{M})$ be a compactly supported smooth function such that $\phi = 1$ on $\text{supp}(\psi)$.

Since D is elliptic, we can construct a parametrix J for it on the open set U by standard methods,

$$JD u = u - H u \quad \forall u \in C_c^\infty(U, \widetilde{\mathcal{E}}|_U)$$

where H has a smooth Schwartz kernel. Define the pseudodifferential operator Q as

$$Q = \sum_{\gamma \in \Gamma} T_\gamma \phi J \psi T_\gamma^* \quad (1)$$

We compute,

$$QDw = \sum_{\gamma \in \Gamma} T_\gamma \phi J \psi D T_\gamma^* w \quad \forall w \in C_c^\infty(\widetilde{M}, \widetilde{\mathcal{E}}), \quad (2)$$

since $T_\gamma D = D T_\gamma \quad \forall \gamma \in \Gamma$. Since D is a 1st order operator, one has

$$D(\psi w) = \psi D w + (D\psi)w$$

so that (2) becomes

$$= \sum_{\gamma \in \Gamma} T_\gamma \phi J D \psi T_\gamma^* w - \sum_{\gamma \in \Gamma} T_\gamma \phi J (D\psi) T_\gamma^* w.$$

Using (1), the expression above becomes

$$= \sum_{\gamma \in \Gamma} T_\gamma \psi T_\gamma^* w - \sum_{\gamma \in \Gamma} T_\gamma \phi H \psi T_\gamma^* w - \sum_{\gamma \in \Gamma} T_\gamma \phi J (D\psi) T_\gamma^* w.$$

Therefore (2) becomes

$$QD = I - R_0$$

where

$$R_0 = \sum_{\gamma \in \Gamma} T_\gamma (\phi H \psi + J(D\psi)) T_\gamma^*$$

has a smooth Schwartz kernel. It is clear from the definition that one has $T_\gamma Q = Q T_\gamma$ and $T_\gamma R_0 = R_0 T_\gamma \quad \forall \gamma \in \Gamma$. Define

$$R_1 = {}^t R_0 + D R_0^t Q - D Q ({}^t R_0).$$

Then $T_\gamma R_1 = R_1 T_\gamma \quad \forall \gamma \in \Gamma$, R_1 has a smooth Schwartz kernel and satisfies

$$DQ = I - R_1.$$

Summarizing, we have the following

Proposition 2.1. *Let M be a compact, good orbifold and $\Gamma \rightarrow \widetilde{M} \rightarrow M$ be the universal orbifold covering space. Let \mathcal{E}, \mathcal{F} be Hermitian vector bundles on M and let $\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}$ be the corresponding lifts to Γ -invariants Hermitian vector bundles on \widetilde{M} . We will assume as before that there is a $(\Gamma, \bar{\sigma})$ -action on $L^2(\widetilde{M})$ given by $T_\gamma = U_\gamma \circ S_\gamma \quad \forall \gamma \in \Gamma$. Then there are $(\Gamma, \bar{\sigma})$ -actions on $L^2(\widetilde{M}, \widetilde{\mathcal{E}})$ and $L^2(\widetilde{M}, \widetilde{\mathcal{F}})$ which are also given by $T_\gamma = U_\gamma \circ S_\gamma \quad \forall \gamma \in \Gamma$.*

Now let $D : L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \rightarrow L^2(\widetilde{M}, \widetilde{\mathcal{F}})$ be a 1st order (Γ, σ) -invariant elliptic operator. Then there is an almost local $(\Gamma, \bar{\sigma})$ -invariant elliptic pseudodifferential operator Q and $(\Gamma, \bar{\sigma})$ -invariant smoothing operators R_0, R_1 which satisfy

$$QD = I - R_0 \quad \text{and} \quad DQ = I - R_1.$$

Define

$$e(D) = \begin{pmatrix} R_0^2 & (R_0 + R_0^2)Q \\ R_1 D & 1 - R_1^2 \end{pmatrix}$$

then $e(D) \in M_2(C_c^\infty(\mathcal{G}, \sigma))$ is an *idempotent*, where $C_c^\infty(\mathcal{G}, \sigma)$ is as defined in section 1. The $C_c^\infty(\mathcal{G}, \sigma)$ -index map is by fiat

$$\text{Ind}_\sigma(D) = [e(D)] - [e_0] \in K_0(C_c^\infty(\mathcal{G}, \sigma))$$

where e_0 is the idempotent

$$e_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is not difficult to see that $\text{Ind}_\sigma(D)$ is independent of the choice of (Γ, σ) -invariant parametrix Q that is needed in its definition.

Lemma 2.2. *Consider the right action of the algebra $C^\infty(M) \otimes \mathbb{C}(\Gamma, \sigma)$ on the vector space of compactly supported smooth functions on \widetilde{M} , $C_c^\infty(\widetilde{M})$ given by*

$$(\xi(f \otimes T_\gamma))(x) = f(p(x))(T_\gamma \xi)(x) \quad \forall \xi \in C_c^\infty(\widetilde{M}), \forall x \in \widetilde{M} \text{ and } \forall \gamma \in \Gamma,$$

where $(T_\gamma \xi)(x) = (U_\gamma S_\gamma \xi)(x)$, $(U_\gamma \xi)(x) = \xi(\gamma x)$ and $(S_\gamma \xi)(x) = e^{i\phi_\gamma(x)} \xi(x)$ for all $\gamma \in \Gamma$. Then $C_c^\infty(\widetilde{M})$ is a finite projective module over $C^\infty(M) \otimes \mathbb{C}(\Gamma, \sigma)$.

Proof. Let $B = C^\infty(M) \otimes \mathbb{C}(\Gamma, \sigma)$ and $E = C_c^\infty(\widetilde{M})$. Let $\{V_i\}_{i=1}^N$ be a finite open cover of M , $\beta_i : V_i \rightarrow \widetilde{M}$ be a smooth section of the orbifold covering $p : \widetilde{M} \rightarrow M$ for $j = 1, \dots, N$. Let $\{\chi_i\}_{i=1}^N$ be a partition of unity of M that is subordinate to the open cover $\{V_i\}_{i=1}^N$ such that the functions $\{\chi_i^{1/2}\}_{i=1}^N$ are smooth on M . For $\xi \in E$, define

$$U \in \text{Hom}_B(E, B^N)$$

by

$$(U\xi)(x, \gamma) = (\chi_1^{1/2}(x)\xi(\gamma^{-1}\beta_1(x)), \dots, \chi_N^{1/2}(x)\xi(\gamma^{-1}\beta_N(x))) \quad \forall x \in M, \forall \gamma \in \Gamma.$$

For $\eta \in B^N$ and $\eta = (\eta_1, \dots, \eta_N)$, define

$$U^* \in \text{Hom}_B(B^N, E)$$

by

$$(U^*\eta)(x) = \sum_{j=1}^N \chi_j^{1/2}(p(x))\eta_j(x, \gamma(\beta_j(p(x)), x)) \quad \forall x \in \widetilde{M}.$$

where $\gamma = \gamma(\beta_j(x), \tilde{x})$ is the unique element in Γ satisfying $\gamma x = \beta_j(p(x))$. It is straightforward to verify that $U^*U = 1_E \in \text{Hom}_B(E, E)$ and therefore one sees that $p_\sigma = UU^* \in \text{Hom}_B(B^N, B^N) = M_N(B)$ is an idempotent. Explicitly, one has

$$(p_\sigma)_{jk} = \chi_j^{1/2} \chi_k^{1/2} \otimes \beta_j \beta_k^{-1} \quad \text{for } 1 \leq j, k \leq N.$$

Therefore $U : E \rightarrow p_\sigma B^N$ is an isomorphism of B -modules, and therefore of $\mathbb{C}(\Gamma, \sigma)$ -modules, where $\mathbb{C}(\Gamma, \sigma)$ is identified with the subalgebra $1 \otimes \mathbb{C}(\Gamma, \sigma)$ of $B = C^\infty(M) \otimes \mathbb{C}(\Gamma, \sigma)$. \square

Using this lemma, we can define a homomorphism θ_σ , defined by $\theta_\sigma(T) = UTU^*$, which transforms (Γ, σ) -invariant linear operators on $C_c^\infty(\widetilde{M})$ into $\mathbb{C}(\Gamma, \sigma)$ -linear endomorphisms of $p_\sigma B^N$. Since $B = C^\infty(M) \otimes \mathbb{C}(\Gamma, \sigma)$, an arbitrary $\mathbb{C}(\Gamma, \sigma)$ -linear endomorphism of $p_\sigma B^N$ is given by a matrix $(S_{i,j})_{1 \leq i,j \leq N}$, where $S_{i,j} \in \text{End}(C^\infty(M)) \otimes \mathbb{C}(\Gamma, \sigma)$.

Let \mathcal{R}_M denote the algebra of all smoothing operators on $C^\infty(M)$.

Proposition 2.3. *The homomorphism θ_σ above, maps the algebra $C_c^\infty(\mathcal{G}, \sigma)$ into $M_N(\mathcal{R}_M \otimes \mathbb{C}(\Gamma, \sigma))$.*

Proof. The map is explicitly given by the equality

$$\theta_\sigma(k)_{ij}(x, y, \gamma) = \chi_i(x)^{1/2} \chi_j(y)^{1/2} k(\beta_j(x), \gamma \beta_j(y))$$

$\forall k \in C_c^\infty(\widetilde{M} \times_\Gamma \widetilde{M}) = C_c^\infty(\mathcal{G}, \sigma)$, $\forall x, y \in M$, $\forall \gamma \in \Gamma$, and therefore $\theta_\sigma(k)_{ij}(\gamma)$ is clearly a smoothing operator. \square

Proposition 2.4. *The homomorphism θ_σ induces a homomorphism*

$$K_0(C_c^\infty(\mathcal{G}, \sigma)) \rightarrow K_0(\mathcal{R}_M \otimes \mathbb{C}(\Gamma, \sigma))$$

that is independent of the choice of $(U_j, \beta_j, \chi_j)_{1 \leq j \leq N}$.

Proof. Suppose that $U' \in \text{Hom}_B(E, B^N)$ is another choice of U . At the expense of replacing N by $2N$, we may assume that there is $W \in GL_N(B)$ such that

$$U' = WU.$$

Then $\theta'_\sigma(T) = U'TU'^* = W\theta_\sigma(T)W^{-1}$ $\forall T \in C_c^\infty(\mathcal{G}, \sigma)$, where W is viewed as a multiplier of the algebra $M_N(\mathcal{R}_M \otimes \mathbb{C}(\Gamma, \sigma))$. Therefore both θ'_σ and θ_σ induce the same map on K -theory. \square

The following lemma is an immediate generalisation of a result in [Co]

Lemma 2.5. *Let M be a compact orbifold of positive dimension. Then there is a canonical isomorphism*

$$\rho : \mathcal{R}_M \rightarrow \mathcal{R}$$

which is unique upto inner automorphisms of \mathcal{R} .

Combining the Proposition 2.4 and Lemma 2.5 above, we obtain a canonical homomorphism in K -theory

$$J : K_0(C_c^\infty(\mathcal{G}, \sigma)) \rightarrow K_0(\mathcal{R}(\Gamma, \sigma))$$

where $\mathcal{R}(\Gamma, \sigma) = \mathcal{R} \otimes \mathbb{C}(\Gamma, \sigma)$, and $J = (\rho \otimes 1) \otimes \theta_\sigma$.

Definition. The (Γ, σ) -index of a (Γ, σ) -invariant elliptic operator $D : L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \rightarrow L^2(\widetilde{M}, \widetilde{\mathcal{F}})$ is defined as

$$\text{Ind}_{(\Gamma, \sigma)}(D) = J(\text{Ind}_\sigma(D)) \in K_0(\mathcal{R}(\Gamma, \sigma))$$

2.2. Heat kernels and the index map. Let D be as before, and for $t > 0$, using the standard off-diagonal estimates for the heat kernel recall that the heat kernels e^{-tD^*D} and e^{-tDD^*} are elements in the $M_N(C_c^\infty(\mathcal{G}, \sigma))$ for some N large enough. Define the idempotent $e_t(D) \in M_{2N}(C_c^\infty(\mathcal{G}, \sigma))$ as follows

$$e_t(D) = \begin{pmatrix} e^{-tD^*D} & e^{-t/2D^*D} \frac{(1-e^{-tD^*D})}{D^*D} D \\ e^{-t/2DD^*} D & 1 - e^{-tDD^*} \end{pmatrix}$$

where f is a smooth, even function on \mathbb{R} which satisfies $f(x)x^2 = e^{-x^2}(1 - e^{-x^2})$.

The relationship with the idempotent $e(D)$ constructed earlier will be explained now. Define for $t > 0$,

$$Q_t = \frac{(1 - e^{-t/2D^*D})}{D^*D} D^*$$

Then one easily verifies that $Q_t D = 1 - e^{-t/2D^*D} = 1 - R_0(t)$ and $DQ_t = 1 - e^{-t/2DD^*} = 1 - R_1(t)$. That is, Q_t is a parametrix for D for all $t > 0$. Therefore one can write

$$e_t(D) = \begin{pmatrix} R_0(t)^2 & (R_0(t) + R_0(t)^2)Q_t \\ R_1(t)D & 1 - R_1(t)^2 \end{pmatrix}.$$

In particular, one has for $t > 0$

$$\text{Ind}_\sigma(D) = [e_t(D)] - [e_0] \in K_0(C_c^\infty(\mathcal{G}, \sigma)).$$

2.3. Twisting an elliptic operator. We will discuss elliptic operators only on good orbifolds, and refer to [Kaw] for the general case. Let M be a good orbifold, that is the universal orbifold cover \widetilde{M} of M is a smooth manifold. Let $\widetilde{W} \rightarrow \widetilde{M}$ be a Γ -invariant Hermitian vector bundle over \widetilde{M} . Let D be a 1st order elliptic differential operator on M ,

$$D : L^2(M, \mathcal{E}) \rightarrow L^2(M, \mathcal{F})$$

acting on L^2 orbifold sections of the orbifold vector bundles \mathcal{E}, \mathcal{F} over M . By fiat, D is a Γ -equivariant 1st order elliptic differential operator \widetilde{D} on the smooth manifold \widetilde{M} ,

$$\widetilde{D} : L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \rightarrow L^2(\widetilde{M}, \widetilde{\mathcal{F}}).$$

Given any connection $\nabla^{\widetilde{W}}$ on \widetilde{W} which is compatible with the Γ action and the Hermitian metric, we wish to define an extension of the elliptic operator \widetilde{D} , to act on sections of $\widetilde{\mathcal{E}} \otimes \widetilde{W}, \widetilde{\mathcal{F}} \otimes \widetilde{W}$.

$$\widetilde{D} \otimes \nabla^{\widetilde{W}} : \Gamma(M, \widetilde{\mathcal{E}} \otimes \widetilde{W}) \rightarrow \Gamma(M, \widetilde{\mathcal{F}} \otimes \widetilde{W})$$

and we want it to satisfy the following property: If \widetilde{W} is a trivial bundle, and ∇^0 is the trivial connection on \widetilde{W} , then for $u \in \Gamma(\widetilde{M}, \widetilde{\mathcal{E}})$, $h \in \Gamma(\widetilde{M}, \widetilde{W})$ such that $\nabla^0 h = 0$,

$$(\widetilde{D} \otimes \nabla^0)(u \otimes h) = (\widetilde{D}u) \otimes h$$

To do this, define a morphism

$$\begin{aligned} S = S_D : \widetilde{\mathcal{E}} \otimes T^* \widetilde{M} &\rightarrow \widetilde{\mathcal{F}} \\ S(u \otimes df) &= \widetilde{D}(fu) - f\widetilde{D}u \end{aligned}$$

for $f \in C^\infty(\widetilde{M})$ and $u \in \Gamma(M, \widetilde{\mathcal{E}})$. Then S is a tensorial. Consider $S = S \otimes 1 : \widetilde{\mathcal{E}} \otimes T^*\widetilde{M} \otimes \widetilde{W} \rightarrow \widetilde{\mathcal{F}} \otimes \widetilde{W}$ defined by

$$S(u \otimes df \otimes e) = S(u \otimes df) \otimes e$$

for u, f as before and $e \in \Gamma(M, \widetilde{W})$.

Recall that a connection $\nabla^{\widetilde{W}}$ on \widetilde{W} is a derivation

$$\nabla^{\widetilde{W}} : \Gamma(\widetilde{M}, \widetilde{W}) \rightarrow \Gamma(\widetilde{M}, T^*\widetilde{M} \otimes \widetilde{W})$$

Define $\widetilde{D} \otimes \nabla^{\widetilde{W}}$ as

$$(\widetilde{D} \otimes \nabla^{\widetilde{W}})(u \otimes e) = (Du) \otimes e + S(u \otimes \nabla^{\widetilde{W}} e)$$

Then $\widetilde{D} \otimes \nabla^{\widetilde{W}}$ is a 1st order elliptic operator.

2.4. Group cocycles and cyclic cocycles. Using the pairing theory of cyclic cohomology and K -theory, due to [Co], we will pair the (Γ, σ) -index of a (Γ, σ) -invariant elliptic operator D on \widetilde{M} with certain cyclic cocycles on $\mathcal{R}(\Gamma, \sigma)$. The cyclic cocycles that we consider come from normalised group cocycles on Γ . More precisely, given a normalised group cocycle $c \in Z^k(\Gamma, \mathbb{C})$, $k = 0, \dots, \dim M$, we define a cyclic cocycle tr_c of dimension k on the twisted group ring $\mathbb{C}(\Gamma, \sigma)$, which is given by

$$\text{tr}_c(a_0 \delta_{g_0}, \dots, a_k \delta_{g_k}) = \begin{cases} a_0 \dots a_k c(g_1, \dots, g_k) \text{tr}(\delta_{g_0} \delta_{g_1} \dots \delta_{g_k}) & \text{if } g_0 \dots g_k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

where $a_j \in \mathbb{C}$ for $j = 0, 1, \dots, k$. To see that this is a cyclic cocycle on $\mathbb{C}(\Gamma, \sigma)$, we first define as done in [Ji], the twisted differential graded algebra $\Omega^\bullet(\Gamma, \sigma)$ as the differential graded algebra of finite linear combinations of symbols

$$g_0 dg_1 \dots dg_n \quad g_i \in \Gamma$$

with module structure and differential given by

$$\begin{aligned} (g_0 dg_1 \dots dg_n)g &= \sum_{j=1}^n (-1)^{n-1} \sigma(g_j, g_{j+1}) g_0 dg_1 \dots d(g_j g_{j+1}) \dots dg_n dg \\ &\quad + (-1)^n \sigma(g_n, g) g_0 dg_1 \dots d(g_n g) \\ d(g_0 dg_1 \dots dg_n) &= dg_0 dg_1 \dots dg_n \end{aligned}$$

We now recall normalised group cocycles. A group k -cocycle is a map $h : \Gamma^{k+1} \rightarrow \mathbb{C}$ satisfying the identities

$$\begin{aligned} h(gg_0, \dots, gg_k) &= h(g_0, \dots, g_k) \\ 0 &= \sum_{i=0}^{k+1} (-1)^i h(g_0, \dots, g_{i-1}, g_{i+1} \dots, g_{k+1}) \end{aligned}$$

Then a normalised group k -cocycle c that is associated to such an h is given by

$$c(g_1, \dots, g_k) = h(1, g_1, g_1 g_2, \dots, g_1 \dots g_k)$$

and it is defined to be zero if either $g_i = 1$ or if $g_1 \dots g_k = 1$. Any normalised group cocycle $c \in Z^k(\Gamma, \mathbb{C})$ determines a k -dimensional cycle via the following closed graded trace on $\Omega^\bullet(\Gamma, \sigma)$

$$\int g_0 dg_1 \dots dg_n = \begin{cases} c(g_1, \dots, g_k) \operatorname{tr}(\delta_{g_0} \delta_{g_1} \dots \delta_{g_k}) & \text{if } n = k \text{ and } g_0 \dots g_k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The higher cyclic trace tr_c is by fiat this closed graded trace.

2.5. Twisted higher index theorem- the cyclic cohomology version. Let M be a compact orbifold of dimension $n = 4\ell$. Let $\Gamma \rightarrow \widetilde{M} \xrightarrow{p} M$ be the universal cover of M and the orbifold fundamental group is Γ . Let D be an elliptic 1st order operator on M and \widetilde{D} be the lift of D to \widetilde{M} ,

$$\widetilde{D} : L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \rightarrow L^2(\widetilde{M}, \widetilde{\mathcal{F}}).$$

Note that \widetilde{D} commutes with the Γ -action on \widetilde{M} .

Now let ω be a closed 2-form on M such that $\widetilde{\omega} = p^*\omega = d\eta$ is *exact*. Define $\nabla = d + i\eta$. Then ∇ is a Hermitian connection on the trivial line bundle over \widetilde{M} , and the curvature of ∇ , $(\nabla)^2 = i\widetilde{\omega}$. (Here $s \in \mathbb{R}$.) Then ∇ defines a projective action of Γ on L^2 spinors as follows: Firstly, observe that since $\widetilde{\omega}$ is Γ -invariant, $0 = \gamma^*\widetilde{\omega} - \widetilde{\omega} = d(\gamma^*\eta - \eta) \ \forall \gamma \in \Gamma$. So $\gamma^*\eta - \eta$ is a closed 1-form on the simply connected manifold \widetilde{M} , therefore

$$\gamma^*\eta - \eta = d\phi_\gamma \quad \forall \gamma \in \Gamma$$

where ϕ_γ is a smooth function on \widetilde{M} satisfying in addition,

- $\phi_\gamma(x) + \phi_{\gamma'}(\gamma x) - \phi_{\gamma'\gamma}(x)$ is independent of $x \in \widetilde{M} \ \forall \gamma, \gamma' \in \Gamma$;
- $\phi_\gamma(x_0) = 0$ for some $x_0 \in \widetilde{M} \ \forall \gamma \in \Gamma$.

Then $\sigma(\gamma, \gamma') = \exp(i\phi_\gamma(\gamma' \cdot x_0))$ defines a multiplier on Γ i.e. $\sigma : \Gamma \times \Gamma \rightarrow U(1)$ satisfies the following identity for all $\gamma, \gamma', \gamma'' \in \Gamma$

$$\sigma(\gamma, \gamma')\sigma(\gamma, \gamma'\gamma'') = \sigma(\gamma\gamma', \gamma'')\sigma(\gamma', \gamma'')$$

For $u \in L^2(\widetilde{M}, \widetilde{\mathcal{E}})$, let $S_\gamma u = e^{i\phi_\gamma} u$ and $U_\gamma u = \gamma^* u$ and $T_\gamma = U_\gamma \circ S_\gamma$ be the composition. Then T defines a projective (Γ, σ) -action on L^2 -spinors, i.e.

$$T_\gamma T_{\gamma'} = \sigma(\gamma, \gamma') T_{\gamma\gamma'}.$$

Consider the *twisted elliptic operator* on \widetilde{M} ,

$$\widetilde{D} \otimes \nabla : L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \rightarrow L^2(\widetilde{M}, \widetilde{\mathcal{F}})$$

Then $\widetilde{D} \otimes \nabla$ no longer commutes with Γ , but it does commute with the projective (Γ, σ) action. Let P_+, P_- be the orthogonal projections onto the nullspace of $\widetilde{D} \otimes \nabla$ and $(\widetilde{D} \otimes \nabla)^*$ respectively since

$$(\widetilde{D} \otimes \nabla) P_+ = 0 \quad \text{and} \quad (\widetilde{D} \otimes \nabla)^* P_- = 0$$

By elliptic regularity, it follows that the Schwartz (or integral) kernels of P_\pm are smooth. Since $\tilde{D} \otimes \nabla$ and its adjoint commutes with the (Γ, σ) action, one has

$$e^{i\phi_\gamma(x)} P_\pm(\gamma x, \gamma y) e^{-i\phi_\gamma(y)} = P_\pm(x, y) \quad \forall \gamma \in \Gamma.$$

In particular, $P_\pm(x, x)$ is a Γ -invariant function on \widetilde{M} . One can define the *von Neumann trace* as Atiyah did in the untwisted case

$$\mathrm{tr}(P_\pm) = \int_M \mathrm{tr}(P_\pm(x, x)) \, dx.$$

The L^2 -index is by definition

$$\mathrm{index}_{L^2}(\tilde{D} \otimes \nabla) = \mathrm{tr}(P_+) - \mathrm{tr}(P_-).$$

To describe the next theorem, we will briefly review some material on characteristic classes for orbifold vector bundles. Let M be a good orbifold, that is the universal orbifold cover $\Gamma \rightarrow \widetilde{M} \rightarrow M$ of M is a smooth manifold. Then the orbifold tangent bundle TM of M , can be viewed as the Γ -equivariant bundle $T\widetilde{M}$ on \widetilde{M} . similar comments apply to the orbifold cotangent bundle T^*M and more generally, any orbifold vector bundle on M . It is then clear that choosing Γ -invariant connections on the Γ -invariant vector bundles on \widetilde{M} , one can define the Chern-Weil representatives of the characteristic classes of the Γ -invariant vector bundles on \widetilde{M} . These characteristic classes are Γ -invariant and so define cohomology classes on M . For further details, see [Kaw].

Theorem 2.6. *Let M be a compact, even dimensional, good orbifold, Γ be its orbifold fundamental group, \tilde{D} be a Γ -invariant elliptic differential operator on \widetilde{M} , where $\Gamma \rightarrow \widetilde{M} \rightarrow M$ is the universal orbifold cover of M . Then for any group cocycle $c \in Z^{2q}(\Gamma)$, $q = 0, 2$, one has*

$$\mathrm{Ind}_{c, \Gamma, \sigma}(\tilde{D} \otimes \nabla) = \frac{q!}{(2\pi i)^q (2q!)} \langle Td(M) \cup ch(symb(D)) \cup f^*(\phi_c) \cup e^\omega, [T^*M] \rangle$$

where $Td(M)$ denotes the Todd characteristic class of the complexified orbifold tangent bundle of M which is pulled back to the orbifold cotangent bundle T^*M , $ch(symb(D))$ is the Chern character of the symbol of the operator D , ϕ_c is the Alexander-Spanier cocycle on $\underline{B}\Gamma$ that corresponds to the group cocycle c and $f : M \rightarrow \underline{B}\Gamma$ is the map that classifies the orbifold universal cover $\widetilde{M} \rightarrow M$, cf. sections 1.7 and 2.5.

Proof. Choose a bounded, almost everywhere smooth Borel cross-section $\beta : M \rightarrow \widetilde{M}$, which can then be used to define the Alexander-Spanier cocycle ϕ_c corresponding to $c \in Z^{2q}(\Gamma)$, and such that $[\phi_c] = [c] \in H^{2q}(\Gamma)$. As in Proposition 2.1, there is an almost local (Γ, σ) -invariant parametrix Q of $\tilde{D} \otimes \nabla$ and (Γ, σ) -invariant smoothing operators R_0, R_1 which satisfy

$$Q(\tilde{D} \otimes \nabla) = I - R_0 \quad \text{and} \quad (\tilde{D} \otimes \nabla)Q = I - R_1.$$

Then as before, one can then construct the index idempotent

$$e(\tilde{D} \otimes \nabla) = \begin{pmatrix} R_0^2 & (R_0 + R_0^2)Q \\ R_1(\tilde{D} \otimes \nabla) & 1 - R_1^2 \end{pmatrix} \in M_2(C_c^\infty(\mathcal{G}, \sigma))$$

and the $C_c^\infty(\mathcal{G}, \sigma)$ -index map is by fiat

$$\mathrm{Ind}_\sigma(\tilde{D} \otimes \nabla) = [e(\tilde{D} \otimes \nabla)] - [e_0] \in K_0(C_c^\infty(\mathcal{G}, \sigma)).$$

where e_0 is the idempotent

$$e_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $R_t = e_t(\tilde{D} \otimes \nabla) - e_0$ and $\theta_\sigma : C_c^\infty(\mathcal{G}, \sigma) \rightarrow \mathbb{C}(\Gamma, \sigma) \otimes \mathcal{L}^2$ be the homomorphism obtained from the section β . Then one has for $t > 0$

$$\text{Ind}_{1,\Gamma,\sigma}(\tilde{D} \otimes \nabla) = \text{tr}(\theta_\sigma(R_t))$$

if $c = 1$ and proceeds exactly as in the case of Atiyah's L^2 index theorem for covering spaces. There is a standard reduction to the case when $D = \partial^\pm \otimes \nabla^\mathcal{E} = \partial_\mathcal{E}^\pm$. Let $k^\pm(t, x, y)$ denote the heat kernel of the lifted Dirac operators $(\tilde{\partial}_\mathcal{E}^\pm \otimes \nabla)^2$ on the universal cover of M , and $P^\pm(x, y)$ the smooth Schwartz kernels of the orthogonal projections P^\pm onto the nullspace of $\tilde{\partial}_\mathcal{E}^\pm \otimes \nabla^s$. By a general result of Cheeger-Gromov-Taylor (see also Roe), the heat kernel $k^\pm(t, x, y)$ converges uniformly over compact subsets of $\widetilde{M} \times \widetilde{M}$ to $P^\pm(x, y)$, as $t \rightarrow \infty$. Therefore one has

$$\begin{aligned} \text{Ind}_{1,\Gamma,\sigma}(\tilde{\partial}_\mathcal{E}^+ \otimes \nabla) &= \text{tr}(\theta_\sigma(R_t)) = \text{tr}_s(e^{-t(\tilde{\partial}_\mathcal{E} \otimes \nabla)^2}) \\ \lim_{t \rightarrow \infty} \text{tr}(e^{-t(\tilde{\partial}_\mathcal{E}^\pm \otimes \nabla)^2}) &= \lim_{t \rightarrow \infty} \int_M \text{tr}(k^\pm(t, x, x)) dx \\ &= \int_M \text{tr}(P^\pm(x, x)) dx \\ &= \text{tr}(P^\pm) \end{aligned} \tag{1}$$

Next observe that

$$\begin{aligned} \frac{\partial}{\partial t} \text{tr}_s(e^{-t(\tilde{\partial}_\mathcal{E} \otimes \nabla)^2}) &= -\text{tr}_s((\tilde{\partial}_\mathcal{E} \otimes \nabla)^2 e^{-t(\tilde{\partial}_\mathcal{E} \otimes \nabla)^2}) \\ &= -\text{tr}_s([\tilde{\partial}_\mathcal{E} \otimes \nabla, (\tilde{\partial}_\mathcal{E} \otimes \nabla) e^{-t(\tilde{\partial}_\mathcal{E} \otimes \nabla)^2}]) \\ &= 0 \end{aligned}$$

since $\tilde{\partial}_\mathcal{E} \otimes \nabla$ is an odd operator. Here tr_s denotes the graded trace, i.e. the composition of the trace tr and the grading operator. Therefore we deduce that

$$\begin{aligned} \text{tr}_s(e^{-t(\tilde{\partial}_\mathcal{E} \otimes \nabla)^2}) &= \lim_{t \rightarrow \infty} \text{tr}_s(e^{-t(\tilde{\partial}_\mathcal{E} \otimes \nabla)^2}) \\ &= \text{tr}_s(P) \\ &= \text{index}_{L^2}(\tilde{\partial}_\mathcal{E}^+ \otimes \nabla). \end{aligned} \tag{2}$$

By the local index theorem of Atiyah-Bott-Patodi [ABP], Getzler [Get], one has

$$\lim_{t \rightarrow 0} (\text{tr}(k^+(t, x, x)) - \text{tr}(k^-(t, x, x))) = [\hat{A}(\Omega) \text{tr}(e^{R^{\tilde{\mathcal{E}}}}) e^\omega]_n \tag{3}$$

where $[\]_n$ denotes the component of degree $n = \dim M$, Ω is the curvature of the metric on \widetilde{M} , $R^{\tilde{\mathcal{E}}}$ is the curvature of the connection on $\tilde{\mathcal{E}}$. Combining equations (1), (2) and (3), one has

$$\begin{aligned} \text{Ind}_{1,\Gamma,\sigma}(\tilde{\partial}_\mathcal{E}^+ \otimes \nabla) &= \text{index}_{L^2}(\tilde{\partial}_\mathcal{E}^+ \otimes \nabla) \\ &= \int_M \hat{A}(\Omega) \text{tr}(e^{R^{\tilde{\mathcal{E}}}}) e^\omega. \end{aligned}$$

We shall now generalize this argument to the case when $c \in Z^{2q}(\Gamma)$, $q > 0$. Here we adapt the strategy and proof in [CM] to our situation. By section 2.2, one has for $t > 0$

$$\begin{aligned} \text{Ind}_{c,\Gamma,\sigma}(\tilde{D} \otimes \nabla) &= \text{tr}_c(\theta_\sigma(R_t), \theta_\sigma(R_t), \dots, \theta_\sigma(R_t)) \\ &= \sum_{\gamma_0 \gamma_1 \dots \gamma_{2q}=1} \text{tr}(\theta_\sigma(R_t)_{\gamma_0} \theta_\sigma(R_t)_{\gamma_1} \dots \theta_\sigma(R_t)_{\gamma_{2q}}) c(1, \gamma_1, \dots, \gamma_1 \gamma_2 \dots \gamma_{2q}) \text{tr}(\delta_{\gamma_0} \dots \delta_{\gamma_{2q}}). \end{aligned}$$

By changing variables, $\gamma_1 = \gamma_1, \gamma_2 = \gamma_1 \gamma_2, \dots, \gamma_{2q} = \gamma_1 \dots \gamma_{2q}$, one obtains

$$\begin{aligned} &\text{Ind}_{c,\Gamma,\sigma}(\tilde{D} \otimes \nabla) \\ &= \sum_{\gamma_1, \gamma_2, \dots, \gamma_{2q} \in \Gamma} \text{tr}(\theta_\sigma(R_t)_{\gamma_1} \theta_\sigma(R_t)_{\gamma_1^{-1} \gamma_2} \dots \theta_\sigma(R_t)_{\gamma_{2q}^{-1}}) c(1, \gamma_1, \dots, \gamma_{2q}) \text{tr}(\delta_{\gamma_1} \delta_{\gamma_1^{-1} \gamma_2} \dots \delta_{\gamma_{2q}^{-1}}) \\ &= \int_{M^{2q+1}} \sum_{\gamma_1, \dots, \gamma_{2q} \in \Gamma} c(1, \gamma_1, \dots, \gamma_{2q}) \text{tr}(\delta_{\gamma_1} \delta_{\gamma_1^{-1} \gamma_2} \dots \delta_{\gamma_{2q}^{-1}}) \times \\ &\quad \text{tr}(R_t(\beta(x_0), \gamma_1 \beta(x_1)) \dots R_t(\gamma_{2q} \beta(x_{2q}), \beta(x_0))) dx_0 \dots dx_{2q} \end{aligned}$$

Observe that if $\beta_U : U \rightarrow \widetilde{M}$ is a smooth local section, then there is a unique element $(\gamma_1, \dots, \gamma_{2q}) \in \Gamma^{2q}$ such that $(\beta(x_0), \gamma_1 \beta(x_1), \dots, \gamma_{2q} \beta(x_{2q})) \in \beta_U(U)^{2q+1}$, and in which case, one has the equality $c(1, \gamma_1, \dots, \gamma_{2q}) = \phi_c(x_0, x_1, \dots, x_{2q})$ (and $\phi_c = 0$ otherwise), where ϕ_c denotes the Γ -equivariant (Alexander-Spanier) $2q$ -cocycle on \widetilde{M} representing the $f^*(c)$ which is the pullback of the group 2-cocycle by the classifying map f , where we have identified M with a fundamental domain for the Γ action on \widetilde{M} , just as was done in the case when the group cocycle $c = 1$. Since R_t is mainly supported near the diagonal as $t \rightarrow 0$, and using the equivariance of R_t , one sees that

$$\text{Ind}_{c,\Gamma,\sigma}(\tilde{D} \otimes \nabla) = \int_{M^{2q+1}} \phi_c(x_0, x_1, \dots, x_{2q}) \text{tr}(R_t(x_0, x_1) \dots R_t(x_{2q}, x_0)) dx_0 dx_1 \dots dx_{2q}$$

The proof is completed by taking the limit as $t \rightarrow 0$ and by applying the *local* higher index Theorems 3.7 and 3.9 in [CM]. \square

Remarks. A particular case of Theorem 2.6 highlights a key new phenomenon in the case of orbifolds, viz. in the special case when the group cocycle $c = 1 \in Z^0(\Gamma)$ is trivial, and when the multiplier $\sigma = 1$ is trivial, then $\text{Ind}_{1,\Gamma,1}(\tilde{D})$ is the L^2 index of \tilde{D} as defined Atiyah. By comparing with the cohomological formula due to Kawasaki [Kaw] for the Fredholm index of the operator D on the orbifold M , we see that in general these are *not* equal, and the error term is a rational number which can be expressed explicitly as a cohomological formula on the lower dimensional strata of the orbifold M . Since Atiyah's L^2 index theorem can be viewed as an integrality statement for the L^2 index of \tilde{D} , $\text{Ind}_{1,\Gamma,1}(\tilde{D})$ in the smooth case, we therefore see that for general orbifolds the L^2 index of \tilde{D} , $\text{Ind}_{1,\Gamma,1}(\tilde{D})$ is only a *rational* number. This was also observed by [Far]. Whereas for smooth two dimensional manifolds the *higher* index associated to the area cocycle is an *integer* [CHMM], in section 5, we show that it is however only a *rational* number for general two dimensional orbifolds. In some other work in progress, we will give an alternate heat kernel proof of a generalization of this theorem, using superconnections.

3. RANGE OF THE TRACE AND THE KADISON CONSTANT

In this section, we will first calculate the range of the canonical trace map on K_0 of the twisted group C^* -algebras for Fuchsian groups Γ of signature (g, ν_1, \dots, ν_n) . We use in an essential way some of the results of the previous section such as the twisted version of the L^2 -index theorem of Atiyah [At], which is due to Gromov [Gr2], and which is proved in Theorem 2.6. This enables us to deduce information about projections in the twisted group C^* -algebras. In the case of no twisting, this follows because the Baum-Connes conjecture is known to be true while these results are also well known for the case of the irrational rotation algebras, and for the twisted groups C^* algebras of the fundamental groups of closed Riemann surfaces of positive genus [CHMM]. Our theorem generalises most of these results. We will apply the results of this section in the next section to study some quantitative aspects of the spectrum of projectively periodic elliptic operators, mainly on orbifold covering spaces of hyperbolic orbifolds.

3.1. The isomorphism classes of algebras $C^*(\Gamma, \sigma)$. Let $\sigma \in Z^2(\Gamma, U(1))$ be a multiplier on Γ , where Γ is a Fuchsian group of signature (g, ν_1, \dots, ν_n) . If $\sigma' \in Z^2(\Gamma, U(1))$ is another multiplier on Γ such that $[\sigma] = [\sigma'] \in H^2(\Gamma, U(1))$, then it can be easily shown that $C^*(\Gamma, \sigma) \cong C^*(\Gamma, \sigma')$. That is, the isomorphism classes of the C^* -algebras $C^*(\Gamma_{g'}, \sigma)$ are naturally parametrized by $H^2(\Gamma, U(1))$. In particular, if we consider only multipliers σ such that $\delta(\sigma) = 0$, we see that these are parametrised by $\ker(\delta) \subset H^2(\Gamma, U(1))$. It follows from the discussion at the beginning of the next subsection that $\ker(\delta) \cong U(1)$. We summarize this below.

Lemma 3.1. *Let Γ be a Fuchsian group of signature (g, ν_1, \dots, ν_n) . Then the isomorphism classes of twisted group C^* -algebras $C^*(\Gamma, \sigma)$ such that $\delta(\sigma) = 0$ are naturally parametrized by $U(1)$.*

3.2. K-theory of twisted group C^* algebras. We begin by computing the K -theory of twisted group C^* -algebras for Fuchsian groups Γ of signature (g, ν_1, \dots, ν_n) . Let σ be a multiplier on Γ . It defines a cohomology class $[\sigma] \in H^2(\Gamma, U(1))$. Consider now the short exact sequence of coefficient groups

$$1 \rightarrow \mathbb{Z} \xrightarrow{i} \mathbb{R} \xrightarrow{e^{2\pi\sqrt{-1}}} U(1) \rightarrow 1,$$

which gives rise to a long exact sequence of cohomology groups (the change of coefficient groups sequence)

$$\dots \rightarrow H^2(\Gamma, \mathbb{Z}) \xrightarrow{i_*} H^2(\Gamma, \mathbb{R}) \xrightarrow{e^{2\pi\sqrt{-1}}_*} H^2(\Gamma, U(1)) \xrightarrow{\delta} H^3(\Gamma, \mathbb{Z}) \xrightarrow{i_*} H^3(\Gamma, \mathbb{R}). \quad (3.1)$$

We first show that the map

$$H^2(\Gamma, U(1)) \xrightarrow{\delta} H^3(\Gamma, \mathbb{Z})$$

is a surjection.

In fact, it is enough to show that $H^3(\Gamma, \mathbb{R}) = \{0\}$. In order to see this it is enough to notice that we have a G action on $B\Gamma_{g'}$ with quotient $B\Gamma$,

$$G \rightarrow B\Gamma_{g'} \xrightarrow{\lambda} B\Gamma \quad (3.2)$$

and therefore the Leray-Serre spectral sequence, we have

$$E^2 = \text{Tor}^{H_*(G, \mathbb{R})}(\mathbb{R}, H_*(B\Gamma_{g'}, \mathbb{R}))$$

that converges to $H_*(B\Gamma, \mathbb{R})$. Moreover, we have

$$E^2 = \text{Tor}^{H_*(G, \mathbb{R})}(\mathbb{R}, \mathbb{R})$$

converging to $H_*(BG, \mathbb{R})$, [McCl] 7.16. Notice also that, with \mathbb{R} coefficients, we have $H_q(BG, \mathbb{R}) = \{0\}$ for $q > 0$. Thus we obtain that, with \mathbb{R} coefficients, $H_q(B\Gamma, \mathbb{R}) \cong H^q(B\Gamma, \mathbb{R})$ is \mathbb{R} in degrees $q = 0$ and $q = 2$, \mathbb{R}^{2g} in degree $q = 1$, and trivial in degrees $q > 2$. In particular, (3.1) now becomes

$$\cdots \rightarrow H^2(\Gamma, \mathbb{Z}) \xrightarrow{i_*} H^2(\Gamma, \mathbb{R}) \xrightarrow{e^{2\pi\sqrt{-1}}_*} H^2(\Gamma, U(1)) \xrightarrow{\delta} H^3(\Gamma, \mathbb{Z}) \xrightarrow{i_*} 0. \quad (3.3)$$

In the following $[\Gamma]$ will denote a choice of a generator in $H_2(B\Gamma, \mathbb{R}) \cong \mathbb{R} \cong H^2(B\Gamma, \mathbb{R})$. Using equation (3.2) and the previous argument, we see that $\lambda_*[\Sigma_{g'}] = \#(G)[\Gamma]$, since $B\Gamma_{g'}$ and $\Sigma_{g'}$ are homotopy equivalent, and where $\#(G)$ denotes the order of the finite group G .

In particular, for any multiplier σ of Γ with $[\sigma] \in H^2(\Gamma_{g'}, U(1))$ and with $\delta(\sigma) = 0$, there is a \mathbb{R} -valued 2-cocycle ζ on Γ with $[\zeta] \in H^2(\Gamma, \mathbb{R})$ such that $[e^{2\pi\sqrt{-1}\zeta}] = [\sigma]$. Define a homotopy $[\sigma_t] = [e^{2\pi\sqrt{-1}t\zeta}] \quad \forall t \in [0, 1]$ which is a homotopy of multipliers σ_t that connects the multiplier σ and the trivial multiplier. Note also that this homotopy is *canonical* and not dependent on the particular choice of ζ . Therefore one obtains a homotopy of isomorphism classes of twisted group C^* -algebras $C^*(\Gamma, \sigma_t)$ connecting $C^*(\Gamma, \sigma)$ and $C^*(\Gamma)$. It is this homotopy which will essentially be used to show that $C^*(\Gamma, \sigma)$ and $C^*(\Gamma)$ have the same K -theory.

Let $\Gamma \subset G$ be a discrete cocompact subgroup of G and A be an algebra admitting an action of Γ by automorphisms. Then the cross product algebra $[A \otimes C_0(G)] \rtimes \Gamma$, is Morita equivalent to the algebra of continuous sections vanishing at infinity $C_0(\Gamma \backslash G, \mathcal{E})$, where $\mathcal{E} \rightarrow \Gamma \backslash G$ is the flat A -bundle defined as the quotient

$$\mathcal{E} = (A \times G)/\Gamma \rightarrow \Gamma \backslash G.$$

Here we consider the diagonal action of Γ on $A \times G$. We refer the reader to [Kas1] for the technical definition of a K -amenable group. However we mention that any solvable Lie group, and in fact any amenable Lie group is K -amenable, and in fact it is shown in [Kas1], [JuKas] that the non-amenable groups $\mathbf{SO}_0(n, 1)$, $\mathbf{SU}(n, 1)$ are K -amenable Lie groups. Also, Cuntz [Cu] has shown that the class of K -amenable groups is closed under the operations of taking subgroups, under free products and under direct products.

Theorem 3.2 ([Kas1], [Kas2]). *If G is K -amenable, then $(A \rtimes \Gamma) \otimes C_0(G)$ and $[A \otimes C_0(G)] \rtimes \Gamma$ have the same K -equivariant K -theory, where K acts in the standard way on G and trivially on the other factors.*

Combining Theorem 3.2 with the remarks above, one gets the following important corollary.

Corollary 3.3. *If G is K -amenable, then $(A \rtimes \Gamma) \otimes C_0(G)$ and $C_0(\Gamma \backslash G, \mathcal{E})$ have the same K -equivariant K -theory. Equivalently, one has for $j = 0, 1$,*

$$K_{Kj}(C_0(\Gamma \backslash G, \mathcal{E})) \cong K_{Kj+\dim(G/K)}(A \rtimes \Gamma).$$

We now come to the main theorem of this section, which generalizes theorems of [CHMM], [PR], [PR2].

Theorem 3.4. *Suppose that Γ is a discrete cocompact subgroup in a K -amenable Lie group G and that K is a maximal compact subgroup of G . Then*

$$K_{\bullet}(C^*(\Gamma, \sigma)) \cong K_K^{\bullet + \dim(G/K)}(\Gamma \backslash G, \delta(B_{\sigma})),$$

where $\sigma \in H^2(\Gamma, U(1))$ is any multiplier on Γ , $K_K^{\bullet}(\Gamma \backslash G, \delta(B_{\sigma}))$ is the twisted K -equivariant K -theory of a continuous trace C^* -algebra B_{σ} with spectrum $\Gamma \backslash G$, and $\delta(B_{\sigma})$ denotes the Dixmier-Douady invariant of B_{σ} .

Proof. Let $\sigma \in H^2(\Gamma, U(1))$, then the twisted cross product algebra $A \rtimes_{\sigma} \Gamma$ is stably equivalent to the cross product $(A \otimes \mathcal{K}) \rtimes \Gamma$ where \mathcal{K} denotes compact operators. This is the Packer-Raeburn stabilization trick [PR], which we now describe in more detail. Let $V : \Gamma \rightarrow U(\ell^2(\Gamma))$ denote the left regular $(\Gamma, \bar{\sigma})$ representation on $\ell^2(\Gamma)$, i.e. for $\gamma, \gamma_1 \in \Gamma$ and $f \in \ell^2(\Gamma)$

$$(V(\gamma_1)f)(\gamma) = \bar{\sigma}(\gamma_1, \gamma_1^{-1}\gamma)f(\gamma_1^{-1}\gamma).$$

Then for $\gamma_1, \gamma_2 \in \Gamma$, V satisfies $V(\gamma_1)V(\gamma_2) = \bar{\sigma}(\gamma_1, \gamma_2)V(\gamma_1\gamma_2)$. That is, V is a projective representation of Γ . Since Ad is trivial on $U(1)$, it follows that $\alpha(\gamma) = Ad(V(\gamma))$ is a representation of Γ on \mathcal{K} . This is easily generalised to the case when \mathbb{C} is replaced by the $*$ algebra A .

Using Corollary 3.3 again, one sees that $A \rtimes_{\sigma} \Gamma \otimes C_0(G)$ and $C_0(\Gamma \backslash G, \mathcal{E}_{\sigma})$ have the same K -equivariant K -theory, whenever G is K -amenable, where

$$\mathcal{E}_{\sigma} = (A \otimes \mathcal{K} \times G)/\Gamma \rightarrow \Gamma \backslash G$$

is a flat $A \otimes \mathcal{K}$ -bundle over $\Gamma \backslash G$ and K is a maximal compact subgroup of G . In the particular case when $A = \mathbb{C}$, one sees that $C_r^*(\Gamma, \sigma) \otimes C_0(G)$ and $C_0(\Gamma \backslash G, \mathcal{E}_{\sigma})$ have the same K -equivariant K -theory whenever G is K -amenable, where

$$\mathcal{E}_{\sigma} = (\mathcal{K} \times G)/\Gamma \rightarrow \Gamma \backslash G.$$

But the twisted K -equivariant K -theory $K_K^*(\Gamma \backslash G, \delta(B_{\sigma}))$ is by definition the K -equivariant K -theory of the continuous trace C^* -algebra $B_{\sigma} = C_0(\Gamma \backslash G, \mathcal{E}_{\sigma})$ with spectrum $\Gamma \backslash G$. Therefore

$$K_{\bullet}(C^*(\Gamma, \sigma)) \cong K_K^{\bullet + \dim(G/K)}(\Gamma \backslash G, \delta(B_{\sigma})).$$

□

Our next main result says that for discrete cocompact subgroups in K -amenable Lie groups, the reduced and unreduced twisted group C^* -algebras have canonically isomorphic K -theories. Therefore all the results that we prove regarding the K -theory of these reduced twisted group C^* -algebras are also valid for the unreduced twisted group C^* -algebras.

Theorem 3.5. *Let $\sigma \in H^2(\Gamma, U(1))$ be a multiplier on Γ and Γ be a discrete cocompact subgroup in a K -amenable Lie group. Then the canonical morphism $C^*(\Gamma, \sigma) \rightarrow C_r^*(\Gamma, \sigma)$ induces an isomorphism*

$$K_*(C^*(\Gamma, \sigma)) \cong K_*(C_r^*(\Gamma, \sigma)).$$

Proof. We note that by the Packer-Raeburn trick, one has

$$C^*(\Gamma, \sigma) \otimes \mathcal{K} \cong \mathcal{K} \rtimes \Gamma$$

and

$$C_r^*(\Gamma, \sigma) \otimes \mathcal{K} \cong \mathcal{K} \rtimes_r \Gamma,$$

where \rtimes_r denotes the reduced crossed product. Since Γ is a lattice in a K -amenable Lie group, the canonical morphism $\mathcal{K} \rtimes \Gamma \rightarrow \mathcal{K} \rtimes_r \Gamma$ induces an isomorphism (cf. [Cu])

$$K_*(\mathcal{K} \rtimes \Gamma) \cong K_*(\mathcal{K} \rtimes_r \Gamma),$$

which proves the result. \square

We now specialize to the case when $G = \mathbf{SO}_0(2, 1)$, $K = \mathbf{SO}(2)$ and $\Gamma = \Gamma(g, \nu_1, \dots, \nu_n)$ is a Fuchsian group, i.e. the orbifold fundamental group of a hyperbolic orbifold of signature (g, ν_1, \dots, ν_n) , $\Sigma(g, \nu_1, \dots, \nu_n)$, where $\Gamma \subset G$ (note that G is K -amenable), or when $G = \mathbb{R}^2$, $K = \{e\}$ and $g = 1$, with Γ being a cocompact crystallographic group.

Proposition 3.6. *Let σ be a multiplier on the Fuchsian group Γ of signature (g, ν_1, \dots, ν_n) such that $\delta(\sigma) = 0$. Then one has*

1. $K_0(C^*(\Gamma, \sigma)) \cong K_0(C^*(\Gamma)) \cong K_{orb}^0(\Sigma(g, \nu_1, \dots, \nu_n)) \cong \mathbb{Z}^{2-n+\sum_{j=1}^n \nu_j}$
2. $K_1(C^*(\Gamma, \sigma)) \cong K_1(C^*(\Gamma)) \cong K_{orb}^1(\Sigma(g, \nu_1, \dots, \nu_n)) \cong \mathbb{Z}^{2g}$.

Proof. Now by a result due to Kasparov [Kas1], which he proves by connecting the regular representation to the trivial one via the complementary series, one has

$$K_\bullet(C^*(\Gamma)) \cong K_{SO(2)}^\bullet(P(g, \nu_1, \dots, \nu_n)) = K_{orb}^\bullet(\Sigma(g, \nu_1, \dots, \nu_n)).$$

We recall next the calculation of Farsi [Far] for the orbifold K -theory of the hyperbolic orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$

$$K_{orb}^0(\Sigma(g, \nu_1, \dots, \nu_n)) \cong K_0(C^*(\Sigma(g, \nu_1, \dots, \nu_n))) = K_{SO(2)}^0(P(g, \nu_1, \dots, \nu_n)) \cong \mathbb{Z}^{2-n+\sum_{j=1}^n \nu_j}$$

and

$$K_{orb}^1(\Sigma(g, \nu_1, \dots, \nu_n)) \cong K_1(C^*(\Sigma(g, \nu_1, \dots, \nu_n))) = K_{SO(2)}^1(P(g, \nu_1, \dots, \nu_n)) \cong \mathbb{Z}^{2g}$$

By Theorem 3.4 we have

$$K_j(C^*(\Gamma)) \cong K_{SO(2)}^j(P(g, \nu_1, \dots, \nu_n)) \quad \text{for } j = 0, 1,$$

and more generally

$$K_j(C^*(\Gamma, \sigma)) \cong K_{SO(2)}^j(P(g, \nu_1, \dots, \nu_n), \delta(B_\sigma)), \quad j = 0, 1,$$

where $B_\sigma = C(P(g, \nu_1, \dots, \nu_n), \mathcal{E}_\sigma)$. Finally, because \mathcal{E}_σ is a locally trivial bundle of C^* -algebras over $P(g, \nu_1, \dots, \nu_n)$, with fibre \mathcal{K} (= compact operators), it has a Dixmier-Douady invariant $\delta(B_\sigma)$ which can be viewed as the obstruction to B_σ being Morita equivalent to $C(\Sigma_g)$. But by assumption $\delta(B_\sigma) = \delta(\sigma) = 0$. Therefore B_σ is Morita equivalent to $C(P(g, \nu_1, \dots, \nu_n))$ and we conclude that

$$K_j(C^*(\Gamma, \sigma)) \cong K_{SO(2)}^j(P(g, \nu_1, \dots, \nu_n)) \cong K_{orb}^j(\Sigma(g, \nu_1, \dots, \nu_n)) \quad j = 0, 1.$$

□

3.3. Twisted Kasparov map. Let Γ be as before, that is, Γ is the orbifold fundamental group of the hyperbolic orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$. Then for any multiplier σ on Γ , the *twisted Kasparov isomorphism*,

$$\mu_\sigma : K_{orb}^\bullet(\Sigma(g, \nu_1, \dots, \nu_n)) \rightarrow K_\bullet(C_r^*(\Gamma, \sigma)) \quad (3.4)$$

is defined as follows. Let $\mathcal{E} \rightarrow \Sigma(g, \nu_1, \dots, \nu_n)$ be an orbifold vector bundle over $\Sigma(g, \nu_1, \dots, \nu_n)$ defining an element $[\mathcal{E}]$ in $K^0(\Sigma(g, \nu_1, \dots, \nu_n))$. As in [Kaw], one can form the twisted Dirac operator $\not\partial_{\mathcal{E}}^+ : L^2(\Sigma(g, \nu_1, \dots, \nu_n), \mathcal{S}^+ \otimes \mathcal{E}) \rightarrow L^2(\Sigma(g, \nu_1, \dots, \nu_n), \mathcal{S}^- \otimes \mathcal{E})$ where \mathcal{S}^\pm denote the $\frac{1}{2}$ spinor bundles over $\Sigma(g, \nu_1, \dots, \nu_n)$. By Proposition 3.2 of the previous subsection, there is a canonical isomorphism

$$K_\bullet(C_r^*(\Gamma, \sigma)) \cong K_{orb}^\bullet(\Sigma(g, \nu_1, \dots, \nu_n)).$$

Both of these maps are assembled to yield the twisted Kasparov map as in (3.4). Observe that $\Sigma(g, \nu_1, \dots, \nu_n) = \underline{B}\Gamma$, and that the twisted Kasparov map has a natural generalisation, which will be studied elsewhere.

We next describe this map more explicitly. One can lift the twisted Dirac operator $\not\partial_{\mathcal{E}}^+$ as above, to a Γ -invariant operator $\widetilde{\not\partial}_{\mathcal{E}}^+$ on $\mathbb{H} = \widetilde{\Sigma}(g, \nu_1, \dots, \nu_n)$, which is the universal orbifold cover of $\Sigma(g, \nu_1, \dots, \nu_n)$,

$$\widetilde{\not\partial}_{\mathcal{E}}^+ : L^2(\mathbb{H}, \widetilde{\mathcal{S}^+ \otimes \mathcal{E}}) \rightarrow L^2(\mathbb{H}, \widetilde{\mathcal{S}^- \otimes \mathcal{E}})$$

Therefore as before in (3.3), for any multiplier σ of Γ with $\delta([\sigma]) = 1$, there is a \mathbb{R} -valued 2-cocycle ζ on Γ with $[\zeta] \in H^2(\Gamma, \mathbb{R})$ such that $[e^{2\pi\sqrt{-1}\zeta}] = [\sigma]$. By the earlier argument using spectral sequences and the fibration as in equation (3.2), we see that the map λ induces an isomorphism $H^2(\Gamma, \mathbb{R}) \cong H^2(\Gamma_{g'}, \mathbb{R})$, and therefore there is a 2-form ω on $\Sigma_{g'}$ such that $[e^{2\pi\sqrt{-1}\omega}] = [\sigma]$. Of course, the choice of ω is not unique, but this will not affect the results that we are concerned with. Let $\tilde{\omega}$ denote the lift of ω to the universal cover \mathbb{H} . Since the hyperbolic plane \mathbb{H} is contractible, it follows that $\tilde{\omega} = d\eta$ where η is a 1-form on \mathbb{H} which is not in general Γ invariant. Now $\nabla = d - i\eta$ is a Hermitian connection on the trivial complex line bundle on \mathbb{H} . Note that the curvature of ∇ is $\nabla^2 = i\tilde{\omega}$. Consider now the twisted Dirac operator $\widetilde{\not\partial}_{\mathcal{E}}^+$ which is twisted again by the connection ∇ ,

$$\widetilde{\not\partial}_{\mathcal{E}}^+ \otimes \nabla : L^2(\mathbb{H}, \widetilde{\mathcal{S}^+ \otimes \mathcal{E}}) \rightarrow L^2(\mathbb{H}, \widetilde{\mathcal{S}^- \otimes \mathcal{E}}).$$

It does not commute with the Γ action, but it does commute with the projective (Γ, σ) -action which is defined by the multiplier σ , and by the twisted L^2 -index theorem of the previous section, it has a Γ - L^2 -index

$$\text{Ind}_{(\Gamma, \sigma)}(\widetilde{\not\partial}_{\mathcal{E}}^+ \otimes \nabla) \in K_0(\mathcal{R}(\Gamma, \sigma)),$$

where as before, \mathcal{R} denotes the algebra of rapidly decreasing sequences on \mathbb{Z}^2 . Then observe that the *twisted Kasparov map* is

$$\mu_\sigma([\mathcal{E}]) = j_*(\text{Ind}_{(\Gamma, \sigma)}(\widetilde{\not\partial}_{\mathcal{E}}^+ \otimes \nabla)) \in K_0(C_r^*(\Gamma, \sigma)),$$

where $j : \mathcal{R}(\Gamma, \sigma) = \mathbb{C}(\Gamma, \sigma) \otimes \mathcal{R} \rightarrow C_r^*(\Gamma, \sigma) \otimes \mathcal{K}$ is the natural inclusion map, and

$$j_* : K_0(\mathcal{R}(\Gamma, \sigma)) \rightarrow K_0(C_r^*(\Gamma, \sigma))$$

is the induced map on K_0 .

The canonical trace on $C_r^*(\Gamma, \sigma)$ induces a linear map

$$[\text{tr}] : K_0(C_r^*(\Gamma, \sigma)) \rightarrow \mathbb{R}$$

which is called the *trace map* in K -theory. Explicitly, first tr extends to matrices with entries in $C^*(\Gamma, \sigma)$ as (with Trace denoting matrix trace):

$$\text{tr}(f \otimes r) = \text{Trace}(r) \text{tr}(f).$$

Then the extension of tr to K_0 is given by $[\text{tr}]([e] - [f]) = \text{tr}(e) - \text{tr}(f)$, where e, f are idempotent matrices with entries in $C^*(\Gamma, \sigma)$.

3.4. Range of the trace map on K_0 . We can now state the major theorem of this section.

Theorem 3.7. *Let Γ be a Fuchsian group of signature (g, ν_1, \dots, ν_n) , and σ be a multiplier of Γ such that $\delta(\sigma) = 0$. Then the range of the trace map is*

$$[\text{tr}](K_0(C_r^*(\Gamma, \sigma))) = \mathbb{Z}\theta + \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i),$$

where $2\pi\theta = \langle [\sigma], [\Gamma] \rangle \in (0, 1]$ is the result of pairing the multiplier σ with the fundamental class of Γ (cf. subsection 3.1).

Proof. We first observe that by the results of the previous subsection the twisted Kasparov map is an isomorphism. Therefore to compute the range of the trace map on K_0 , it suffices to compute the range of the trace map on elements of the form

$$\mu_\sigma([\mathcal{E}^0] - [\mathcal{E}^1])$$

for any element

$$[\mathcal{E}^0] - [\mathcal{E}^1] \in K_{orb}^0(\Sigma(g, \nu_1, \dots, \nu_n)).$$

where $\mathcal{E}^0, \mathcal{E}^1$ are orbifold vector bundles over the orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$, which as in section 1, can be viewed as G -equivariant vector bundles over the Riemann surface $\Sigma_{g'}$ which is an orbifold G covering of the orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$.

By the twisted L^2 index theorem for orbifolds from the previous section, one has

$$[\text{tr}](\text{Ind}_{(\Gamma, \sigma)}(\widetilde{\partial_{\mathcal{E}}^+} \otimes \nabla)) = \frac{1}{2\pi} \int_{\Sigma(g, \nu_1, \dots, \nu_n)} \hat{A}(\Omega) \text{tr}(e^{R^{\mathcal{E}}}) e^{\omega}. \quad (3.5)$$

We next simplify the right hand side of equation (3.5) using

$$\begin{aligned} \hat{A}(\Omega) &= 1 \\ \text{tr}(e^{R^{\mathcal{E}}}) &= \text{rank } \mathcal{E} + \text{tr}(R^{\mathcal{E}}) \\ e^{\omega} &= 1 + \omega. \end{aligned}$$

Therefore one has

$$[\text{tr}](\text{Ind}_{(\Gamma, \sigma)}(\widetilde{\partial_{\mathcal{E}}^+} \otimes \nabla)) = \text{rank } \mathcal{E} \frac{\int_{\Sigma(g, \nu_1, \dots, \nu_n)} \omega}{2\pi} + \frac{\int_{\Sigma(g, \nu_1, \dots, \nu_n)} \text{tr}(R^{\mathcal{E}})}{2\pi},$$

Now by the index theorem for orbifolds, due to Kawasaki [Kaw], we see that

$$\frac{\int_{\Sigma(g, \nu_1, \dots, \nu_n)} \text{tr}(R^\mathcal{E})}{2\pi} + \frac{\sum_{i=1}^n \beta_i / \nu_i}{2\pi} = \text{index}(\partial_\mathcal{E}^+) \in \mathbb{Z},$$

Therefore we see that

$$\frac{\int_{\Sigma(g, \nu_1, \dots, \nu_n)} \text{tr}(R^\mathcal{E})}{2\pi} \in \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i)$$

Observe that

$$\int_{\Sigma(g, \nu_1, \dots, \nu_n)} \omega = \frac{1}{\#(G)} \int_{\Sigma_{g'}} \omega = \langle [\omega], [\Gamma] \rangle$$

since $\Sigma_{g'}$ is an orbifold G covering of the orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$ and $[\Gamma]$ is equal to $\frac{[\Sigma_{g'}]}{\#(G)}$, cf. section 3.1 and that by assumption,

$$\frac{\langle [\omega], [\Gamma] \rangle}{2\pi} - \theta \in \mathbb{Z}.$$

It follows that the range of the trace map on K_0 is $\mathbb{Z} \frac{\langle [\omega], [\Gamma] \rangle}{2\pi} + \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i) = \mathbb{Z}\theta + \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i)$. \square

We will now discuss one application of this result, leaving further applications to the next section. The application studies the number of projections in the twisted group C^* -algebra, which is a problem of independent interest.

Proposition 3.8. *Let σ be a multiplier on Γ such that $\delta(\sigma) = 0$, and $2\pi\theta = \langle [\sigma], [\Gamma] \rangle \in (0, 1]$ be the result of pairing the cohomology class of σ with the fundamental class of Γ . If θ is rational, then there are at most a finite number of unitary equivalence classes of projections, other than 0 and 1, in the reduced twisted group C^* -algebra $C_r^*(\Gamma, \sigma)$.*

Proof. By assumption, $\theta = p/q$. Let P be a projection in $C_r^*(\Gamma, \sigma)$. Then $1 - P$ is also a projection in $C_r^*(\Gamma, \sigma)$ and one has

$$1 = \text{tr}(1) = \text{tr}(P) + \text{tr}(1 - P).$$

Each term in the above equation is non-negative. Since σ is rational and by Theorem 3.3, it follows that the Kadison constant $C_\sigma(\Gamma) > 0$ (see section 4 for the definition) and $\text{tr}(P) \in \{0, C_\sigma(\Gamma), 2C_\sigma(\Gamma), \dots, 1\}$. By faithfulness and normality of the trace tr , it follows that there are at most a finite number of unitary equivalence classes of projections, other than those of 0 and 1 in $C_r^*(\Gamma, \sigma)$. \square

4. APPLICATIONS TO THE SPECTRAL THEORY OF PROJECTIVELY PERIODIC ELLIPTIC OPERATORS AND THE CLASSIFICATION OF TWISTED GROUP C^* ALGEBRAS

In this section, we apply the range of the trace theorem of section 3, to prove some qualitative results on the spectrum of projectively periodic self adjoint elliptic operators on the universal covering of a good orbifold. In particular, we study generalizations of the hyperbolic analogues of the Ten Martini Problem in [CHMM] and the Bethe-Sommerfeld conjecture. We also classify upto isomorphism, the twisted group C^* algebras for a cocompact Fuchsian group.

Let M be a compact, good orbifold, that is, the universal cover $\Gamma \rightarrow \widetilde{M} \rightarrow M$ is a smooth manifold and we will assume as before that there is a $(\Gamma, \bar{\sigma})$ -action on $L^2(\widetilde{M})$ given by $T_\gamma = U_\gamma \circ S_\gamma \forall \gamma \in \Gamma$. Let $\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}$ be Hermitian vector bundles on M and let $\widetilde{\mathcal{E}}, \widetilde{\mathcal{F}}$ be the corresponding lifts to Γ -invariants Hermitian vector bundles on \widetilde{M} . Then there are (Γ, σ) -actions on $L^2(\widetilde{M}, \widetilde{\mathcal{E}})$ and $L^2(\widetilde{M}, \widetilde{\mathcal{F}})$ which are also given by $T_\gamma = U_\gamma \circ S_\gamma \forall \gamma \in \Gamma$.

Now let $D : L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \rightarrow L^2(\widetilde{M}, \widetilde{\mathcal{F}})$ be a self adjoint elliptic differential operator that commutes with the $(\Gamma, \bar{\sigma})$ -action that was defined earlier. We begin with some basic facts about the spectrum of such an operator. Recall that the *discrete spectrum* of D , $\text{spec}_{\text{disc}}(D)$ consists of all the eigenvalues of D that have finite multiplicity, and the *essential spectrum* of D , $\text{spec}_{\text{ess}}(D)$ consists of the complement $\text{spec}(D) \setminus \text{spec}_{\text{disc}}(D)$. That is, $\text{spec}_{\text{ess}}(D)$ consists of the set of accumulation points of the spectrum of D , $\text{spec}(D)$. Our first goal is to prove that the essential spectrum is unbounded. Our proof will be a modification of an argument in [Sar].

Lemma 4.1. *Let $D : L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \rightarrow L^2(\widetilde{M}, \widetilde{\mathcal{F}})$ be a self adjoint elliptic differential operator that commutes with the $(\Gamma, \bar{\sigma})$ -action. Then the discrete spectrum of D is empty.*

Proof. Let λ be an eigenvalue of D and V denote the corresponding eigenspace. Then V is a (Γ, σ) -invariant subspace of $L^2(\widetilde{M}, \widetilde{\mathcal{E}})$. If \mathcal{F} is a fundamental domain for the action of Γ on \widetilde{M} , one sees that

$$L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \cong L^2(\Gamma) \otimes L^2(\mathcal{F}, \widetilde{\mathcal{E}}|_{\mathcal{F}})$$

which can be proved by choosing a bounded measurable almost everywhere smooth section of the orbifold covering $\widetilde{M} \rightarrow M$. Here $(\Gamma, \bar{\sigma})$ -action on $L^2(\mathcal{F}, \widetilde{\mathcal{E}}|_{\mathcal{F}})$ is trivial, and is the left regular $(\Gamma, \bar{\sigma})$ -action on $L^2(\Gamma)$. Therefore it suffices to show that the dimension of *any* $(\Gamma, \bar{\sigma})$ -invariant subspace V of $L^2(\Gamma)$ is infinite dimensional. Let $\{v_1, \dots, v_N\}$ be an orthonormal basis for V . Then one has

$$T_\gamma v_i(\gamma') = \sum_{j=1}^N U_{ij}(\gamma) v_j(\gamma') \quad \forall \gamma, \gamma' \in \Gamma$$

where $U = (U_{ij}(\gamma))$ is some $N \times N$ unitary matrix. Therefore

$$\begin{aligned} N &= \sum_{j=1}^N \|v_j\|^2 = \sum_{j=1}^N \sum_{\gamma \in \Gamma} |v_j(\gamma)|^2 \\ &= \sum_{\gamma \in \Gamma} \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N U_{ij}(\gamma) \overline{U_{ik}(\gamma)} v_j(\gamma) \overline{v_k(\gamma)} \\ &= \sum_{\gamma \in \Gamma} \sum_{j=1}^N |v_j(\gamma)|^2 \\ &= \#(\Gamma) \sum_{j=1}^N |v_j(\gamma)|^2. \end{aligned}$$

Since $\#(\Gamma) = \infty$, it follows that either $N = 0$ or $N = \infty$. □

Corollary 4.2. *Let D be as in Lemma 4.1 above. Then the essential spectrum of D coincides with the spectrum of D , and so it is unbounded.*

Proof. By the Lemma above, we conclude that $\text{spec}_{\text{ess}}(D)$ and $\text{spec}(D)$ coincide. Since D is an unbounded self-adjoint operator, it is a standard fact that $\text{spec}(D)$ is unbounded cf. [Gi], yielding the result. \square

Note that in general the spectral projections of D , $E_\lambda \notin C^*(\mathcal{G}, \sigma)$. However one has

Proposition 4.3 (Sunada, Bruning-Sunada). *Let D be as in Lemma 4.1 above. If $\lambda_0 \notin \text{spec}(D)$, then $E_{\lambda_0} \in C^*(\mathcal{G}, \sigma)$.*

Proof. Firstly, there is a standard reduction to the case when D is positive and of even order $d \geq 2$ cf. [BrSu], so we will assume this without loss of generality. By a result of Greiner, see also Bruning-Sunada [BrSu], there are off diagonal estimates for the Schwartz kernel of the heat operator e^{-tD}

$$|k_t(x, y)| \leq C_1 t^{-n/d} \exp\left(-C_2 d(x, y)^{d/(d-1)} t^{-1/(d-1)}\right)$$

for some positive constants C_1, C_2 and for $t > 0$ in any compact interval. Since the volume growth of a orbifold covering space is at most exponential, we see in particular that $|k_t(x, y)|$ is L^1 in both variable separately, so that

$$e^{-tD} \in C^*(\mathcal{G}, \sigma).$$

Note that $\chi_{[0, e^{-t\lambda_1}]}(D) = \chi_{[0, \lambda_1]}(e^{-tD})$ Let $t = 1$ and $\lambda_1 = -\log \lambda_0$. Then $\lambda_1 \notin \text{spec}(e^{-D})$ and

$$\chi_{[0, \lambda_1]}(e^{-D}) = \phi(e^{-D})$$

where ϕ is a compactly supported smooth function, $\phi \cong 1$ on $[0, \lambda_1]$ and $\phi \cong 0$ on the remainder of the spectrum. Since $C^*(\mathcal{G}, \sigma)$ is closed under the continuous functional calculus, it follows that $\phi(e^{-D}) \in C^*(\mathcal{G}, \sigma)$, that is $E_{\lambda_0} \in C^*(\mathcal{G}, \sigma)$. \square

We will now recall the definition of the Kadison constant of a twisted group C^* -algebra. The Kadison constant of $C_r^*(\Gamma, \sigma)$ is defined by:

$$C_\sigma(\Gamma) = \inf\{\text{tr}(P) : P \text{ is a non-zero projection in } C_r^*(\Gamma, \sigma) \otimes \mathcal{K}\}.$$

Proposition 4.4. *Let Γ be a Fuchsian group of signature (g, ν_1, \dots, ν_n) . Let σ be a multiplier on Γ such that $\delta(\sigma) = 0$, and $2\pi\theta = \langle [\sigma], [\Gamma] \rangle \in (0, 1]$ be the result of pairing the cohomology class of σ with the fundamental class of Γ . If θ is rational, then the spectrum of any $(\Gamma, \bar{\sigma})$ -invariant elliptic differential operator D on \widetilde{M} has a band structure, in the sense that the intersection of the resolvent set with any compact interval in \mathbb{R} has only a finite number of components. Here $\Gamma \rightarrow \widetilde{M} \rightarrow M$ is the universal orbifold covering of a compact good orbifold M with orbifold fundamental group Γ . In particular, the intersection of $\text{spec}(D)$ with any compact interval in \mathbb{R} is never a Cantor set.*

Proof. By Proposition 4.1 and Theorem 3.3, it follows that one has the estimate $C_\sigma(\Gamma) \geq 1/q > 0$. Then one applies the main result in Brüning-Sunada [BrSu] to deduce the proposition. \square

In words, we have shown that whenever the multiplier is rational, then the spectrum of a projectively periodic elliptic operator is the union of countably many (possibly degenerate) closed intervals, which can only accumulate at infinity.

Recall the important Γ -invariant elliptic differential operator, which is called the Schrödinger operator

$$H_V = \Delta + V$$

where Δ denotes the Laplacian on functions on \widetilde{M} and V is a Γ -invariant function on \widetilde{M} . It is known that the Baum-Connes conjecture is true for all amenable discrete subgroups of a connected Lie group and also for discrete subgroups of $SO(n, 1)$ [Kas1] and $SU(n, 1)$ [JuKas]. For all these groups Γ , it follows that the Kadison constant $C_1(\Gamma)$ is positive. Therefore we see by the arguments above that the spectrum of the periodic elliptic operator H_V is the union of countably many (possibly degenerate) closed intervals, which can only accumulate at infinity. This gives evidence for the following

Conjecture (The Generalized Bethe-Sommerfeld conjecture). *The spectrum of any Γ -invariant Schrödinger operator H_V has only a finite number of bands, in the sense that the intersection of the resolvent set with \mathbb{R} has only a finite number of components.*

We remark that the Bethe-Sommerfeld conjecture has been proved completely by Skripanov [Skri] in the Euclidean case.

This leaves open the question of whether there are $(\Gamma, \bar{\sigma})$ -invariant elliptic differential operators D on \mathbb{H} with Cantor spectrum when θ is irrational. In the Euclidean case, this is usually known as the *Ten Martini Problem*, and is to date, not completely solved, though much progress has been made (cf. [Sh]). We pose a generalization of this problem to the hyperbolic case (which also includes the Euclidean case):

Conjecture (The Generalized Ten Dry Martini Problem). *Let σ be a multiplier on Γ such that $\delta(\sigma) = 0$, and $2\pi\theta = \langle [\sigma], [\Gamma] \rangle \in (0, 1]$ be the result of pairing the cohomology class of σ with the fundamental class of Γ . If θ is irrational, then there is a $(\Gamma, \bar{\sigma})$ -invariant elliptic differential operator D on \mathbb{H} which has a Cantor set type spectrum, in the sense that the intersection of $\text{spec}(D)$ with some compact interval in \mathbb{R} is a Cantor set.*

4.1. On the classification of twisted group C^* -algebras. We will now use the range of the trace Theorem 3.3, to give a complete classification, up to isomorphism, of the twisted group C^* -algebras $C^*(\Gamma, \sigma)$, where we assume as before that $\delta(\sigma) = 0$.

Proposition 4.5 (Isomorphism classification of twisted group C^* -algebras). *Let $\sigma, \sigma' \in H^2(\Gamma, \mathbb{R}/\mathbb{Z})$ be multipliers on Γ satisfying $\delta(\sigma) = 0 = \delta(\sigma')$, and $2\pi\theta = \langle \sigma, [\Gamma] \rangle \in (0, 1]$, $2\pi\theta' = \langle \sigma', [\Gamma] \rangle \in (0, 1]$ be the result of pairing σ, σ' with the fundamental class of Γ . Then $C^*(\Gamma, \sigma) \cong C^*(\Gamma, \sigma')$ if and only if $\theta' \in \{(\theta + \sum_{i=1}^n \beta_i/\nu_i) \bmod 1, (1 - \theta + \sum_{i=1}^n \beta_i/\nu_i) \bmod 1\}$, where $0 \leq \beta_i \leq \nu_i - 1 \quad \forall i = 1, \dots, n$.*

Proof. Let tr and tr' denote the canonical traces on $C^*(\Gamma, \sigma)$ and $C^*(\Gamma, \sigma')$ respectively. Let

$$\phi : C^*(\Gamma, \sigma) \cong C^*(\Gamma, \sigma')$$

be an isomorphism, and let

$$\phi_* : K_0(C^*(\Gamma, \sigma)) \cong K_0(C^*(\Gamma, \sigma'))$$

denote the induced map on K_0 . By Theorem 3.3, the range of the trace map on K_0 is

$$[\text{tr}](K_0(C^*(\Gamma, \sigma))) = \mathbb{Z}\theta + \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i)$$

and

$$[\text{tr}'](K_0(C^*(\Gamma, \sigma'))) = \mathbb{Z}\theta' + \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i).$$

Therefore if θ is irrational, then $\mathbb{Z}\theta + \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i) = \mathbb{Z}\theta' + \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i)$ implies that θ' is also irrational and that $\theta \pm \theta' \in \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i)$. Since $\theta, \theta' \in (0, 1]$, one deduces that $\theta' \in \{(\theta + \sum_{i=1}^n \beta_i/\nu_i) \bmod 1, (1 - \theta + \sum_{i=1}^n \beta_i/\nu_i) \bmod 1\}$, where $0 \leq \beta_i \leq \nu_i - 1 \quad \forall i = 1, \dots, n$. Virtually the same argument holds when θ is rational, but one argues in K -theory first, and applies the trace only at the final step.

First observe that a diffeomorphism $C : \Sigma_{g'} \rightarrow \Sigma_{g'}$ lifts to a diffeomorphism C' of \mathbb{H} such that $C'\Gamma C'^{-1} = \Gamma$, i.e. it defines an automorphism of Γ . Recall that the finite group

$$G = \{C_i : C_i^{\nu_i} = 1 \quad \forall i = 1, \dots, n\}$$

acts on $\Sigma_{g'}$ with quotient $\Sigma(g, \nu_1, \dots, \nu_n)$. By the observation above, we see that G also acts as automorphisms of Γ . Now $C_i[\Gamma] = \lambda_i[\Gamma]$, where $\lambda_i \in \mathbb{C}$. Since $C_i C_j[\Gamma] = \lambda_i \lambda_j[\Gamma]$ and $C_i^{\nu_i} = 1$, it follows that λ_i is an ν_i^{th} root of unity, i.e. $\lambda_i = e^{2\pi\sqrt{-1}(1/\nu_i)}$. Let $C \in G$, i.e. $C = \prod_{i=1}^n C_i^{\beta_i}$. We evaluate $\langle C^*[\sigma], [\Gamma] \rangle = \langle [\sigma], C_*[\Gamma] \rangle = \langle [\prod_{i=1}^n \lambda_i^{\beta_i} \sigma], [\Gamma] \rangle = \theta + \sum_{i=1}^n \beta_i/\nu_i$. As in subsection 3.1 we see that $C^*[\sigma] = [\prod_{i=1}^n \lambda_i^{\beta_i} \sigma] \in \ker \delta \subset H^2(\Gamma, U(1))$. Therefore the automorphism C_* of Γ induces an isomorphism of twisted group C^* -algebras

$$C^*(\Gamma, \sigma) \cong C^*(\Gamma, C^* \sigma) \cong C^*(\Gamma, \lambda \sigma).$$

where $\lambda = \prod_{i=1}^n \lambda_i^{\beta_i}$.

Now let $\psi : \Sigma_{g'} \rightarrow \Sigma_{g'}$ be an orientation reversing diffeomorphism. Then as observed earlier, ψ induces an automorphism $\psi_* : \Gamma \rightarrow \Gamma$ of Γ . We evaluate $\langle \psi^*[\sigma], [\Gamma] \rangle = \langle [\sigma], \psi_*[\Gamma] \rangle = \langle [\sigma], [\Gamma] \rangle = \theta$, since ψ is orientation reversing. As in subsection 3.1 we see that $\psi^*[\sigma] = [\bar{\sigma}] \in \ker \delta \subset H^2(\Gamma, U(1))$. Therefore the automorphism ψ_* of Γ induces an isomorphism of twisted group C^* -algebras

$$C^*(\Gamma, \sigma) \cong C^*(\Gamma, \psi^* \sigma) \cong C^*(\Gamma, \bar{\sigma}).$$

Therefore if $\theta' \in \{(\theta + \sum_{i=1}^n \beta_i/\nu_i) \bmod 1, (1 - \theta + \sum_{i=1}^n \beta_i/\nu_i) \bmod 1\}$, where $0 \leq \beta_i \leq \nu_i - 1 \quad \forall i = 1, \dots, n$, one has $C^*(\Gamma, \sigma) \cong C^*(\Gamma, \sigma')$, completing the proof of the proposition. \square

5. RANGE OF THE HIGHER TRACE ON K -THEORY

In this section, we compute the range of the 2-trace tr_c on K -theory of the twisted group C^* algebra, where c is a 2-cocycle on the group, generalising the work of [CHMM]. Suppose as before that Γ is a discrete, cocompact subgroup of $PSL(2, \mathbb{R})$ of signature (g, ν_1, \dots, ν_n) . That is, Γ is the orbifold fundamental group of a compact hyperbolic orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$ of signature (g, ν_1, \dots, ν_n) . Then for any multiplier σ on Γ such that $\delta(\sigma) = 0$, one has the *twisted Kasparov isomorphism*,

$$\mu_\sigma : K_{orb}^\bullet(\Sigma(g, \nu_1, \dots, \nu_n)) \rightarrow K_\bullet(C_r^*(\Gamma, \sigma))$$

is defined as in the previous section. We note in the following section (using a result of [Ji]) that given any projection P in $C_r^*(\Gamma, \sigma)$ there is both a projection \tilde{P} in the same K_0 class but lying in a dense subalgebra, stable under the holomorphic functional calculus, and a Fredholm module for this dense subalgebra, which may be paired with \tilde{P} to obtain an analytic index. On the other hand, by the results of the current section, given any such projection P there is a topological index that we can associate to it. The main result we prove here is that the range of the 2-trace tr_c on K -theory of the twisted group C^* algebra is always an integer multiple of a rational number. This will enable us to compute the range of values of the Hall conductance in the quantum Hall effect on hyperbolic space, generalising the results in [CHMM].

The first step in the proof is to show that given an additive group cocycle $c \in Z^2(\Gamma)$ we may define canonical pairings with $K^0(\Sigma(g, \nu_1, \dots, \nu_n))$ and $K_0(C_r^*(\Gamma, \sigma))$ which are related by the twisted Kasparov isomorphism, by generalizing some of the results of Connes and Connes-Moscovici to the twisted case. The group 2-cocycle c may be regarded as a skew symmetrised function on $\Gamma \times \Gamma \times \Gamma$, so that we can use the results in section 2 to obtain a cyclic 2-cocycle tr_c on $\mathbb{C}(\Gamma, \sigma) \otimes \mathcal{R}$ by defining:

$$\text{tr}_c(f^0 \otimes r^0, f^1 \otimes r^1, f^2 \otimes r^2) = \text{Tr}(r^0 r^1 r^2) \sum_{g_0 g_1 g_2 = 1} f^0(g_0) f^1(g_1) f^2(g_2) c(1, g_1, g_1 g_2) \sigma(g_1, g_2).$$

Note that tr_c extends to $\mathbb{C}(\Gamma, \sigma) \otimes \mathcal{L}^2$, (where \mathcal{L}^2 denotes Hilbert-Schmidt operators) and since $\mathbb{C}(\Gamma, \sigma) \otimes \mathcal{R} \subset \mathbb{C}(\Gamma, \sigma) \otimes \mathcal{L}^2$ by the pairing theory of [Co] one gets an additive map

$$[\text{tr}_c] : K_0(\mathcal{R}(\Gamma, \sigma)) \rightarrow \mathbb{R}.$$

Explicitly, $[\text{tr}_c]([e] - [f]) = \widetilde{\text{tr}}_c(e, \dots, e) - \widetilde{\text{tr}}_c(f, \dots, f)$, where e, f are idempotent matrices with entries in $(\mathcal{R}(\Gamma, \sigma))^\sim =$ unital algebra obtained by adding the identity to $\mathcal{R}(\Gamma, \sigma)$ and $\widetilde{\text{tr}}_c$ denotes the canonical extension of tr_c to $(\mathcal{R}(\Gamma, \sigma))^\sim$. Let $\widetilde{\partial_{\mathcal{E}}^+} \otimes \nabla$ be the Dirac operator defined in the previous section, which is invariant under the projective action of the fundamental group defined by σ . Recall that by definition, the (c, Γ, σ) -index of $\widetilde{\partial_{\mathcal{E}}^+} \otimes \nabla$ is

$$\text{Ind}_{c, \Gamma, \sigma}(\widetilde{\partial_{\mathcal{E}}^+} \otimes \nabla) = [\text{tr}_c](\text{Ind}_{(\Gamma, \sigma)}(\widetilde{\partial_{\mathcal{E}}^+} \otimes \nabla)) = \langle [\text{tr}_c], \mu_\sigma([\mathcal{E}]) \rangle \in \mathbb{R}.$$

It only depends on the cohomology class $[c] \in H^2(\Gamma)$, and it is linear with respect to $[c]$. We assemble this to give the following theorem.

Theorem 5.1. *Given $[c] \in H^2(\Gamma)$ and $\sigma \in H^2(\Gamma, U(1))$ a multiplier on Γ , there is a canonical additive map*

$$\langle [c], \cdot \rangle : K_{orb}^0(\Sigma(g, \nu_1, \dots, \nu_n)) \rightarrow \mathbb{R},$$

which is defined as

$$\langle [c], [\mathcal{E}] \rangle = \text{Ind}_{c, \Gamma, \sigma}(\widetilde{\partial_{\mathcal{E}}^+} \otimes \nabla) = [\text{tr}_c](\text{Ind}_{(\Gamma, \sigma)}(\widetilde{\partial_{\mathcal{E}}^+} \otimes \nabla)) = \langle [\text{tr}_c], \mu_{\sigma}([\mathcal{E}]) \rangle \in \mathbb{R}.$$

Moreover, it is linear with respect to $[c]$.

The *area cocycle* c of the Fuchsian group Γ is a canonically defined 2-cocycle on Γ that is defined as follows. Firstly, recall that there is a well known area 2-cocycle on $PSL(2, \mathbb{R})$ cf. [Co2] defined as follows: $PSL(2, \mathbb{R})$ acts on \mathbb{H} such that $\mathbb{H} \cong PSL(2, \mathbb{R})/SO(2)$. Then $c(g_1, g_2) = \text{Area}(\Delta(o, g_1.o, g_2^{-1}.o)) \in \mathbb{R}$, where o denotes an origin in \mathbb{H} and $\text{Area}(\Delta(a, b, c))$ denotes the hyperbolic area of the geodesic triangle in \mathbb{H} with vertices at $a, b, c \in \mathbb{H}$. Then the restriction of c to the subgroup Γ is the area cocycle c of Γ .

Corollary 5.2. *Let $c, [c] \in H^2(\Gamma)$, be the area cocycle. Then one has*

$$\langle [c], [\mathcal{E}] \rangle = \phi \text{rank } \mathcal{E} \in \phi \mathbb{Z}.$$

where $\phi = 2(g-1) + (n-\nu) \in \mathbb{Q}$ and $\nu = \sum_{j=1}^n 1/\nu_j$.

Proof. By our generalization of the Connes-Moscovici higher index theorem [CM] to the twisted case of elliptic operators on an orbifold covering space and that are invariant under the projective action of the orbifold fundamental group defined by σ , cf. section 2, one has

$$[\text{tr}_c](\text{Ind}_{(\Gamma, \sigma)}(\widetilde{\partial_{\mathcal{E}}^+} \otimes \nabla)) = \frac{1}{2\pi\#(G)} \int_{\Sigma_{g'}} \hat{A}(\Omega) \text{tr}(e^{R^{\mathcal{E}}}) e^{\omega} \psi^*(\tilde{c}), \quad (5.1)$$

where $\psi : \Sigma_{g'} \rightarrow \Sigma_{g'}$ is the lift of the map $f : \Sigma(g, \nu_1, \dots, \nu_n) \rightarrow \Sigma(g, \nu_1, \dots, \nu_n)$ (since $\underline{B}\Gamma = \Sigma(g, \nu_1, \dots, \nu_n)$ in this case) which is the classifying map of the orbifold universal cover (and which in this case is the identity map) and $[\tilde{c}]$ degree 2 cohomology class on $\Sigma_{g'}$ that is the lift of c to $\Sigma_{g'}$. We next simplify the right hand side of (5.1) using the fact that $\hat{A}(\Omega) = 1$ and that

$$\begin{aligned} \text{tr}(e^{R^{\mathcal{E}}}) &= \text{rank } \mathcal{E} + \text{tr}(R^{\mathcal{E}}), \\ \psi^*(\tilde{c}) &= \tilde{c}, \\ e^{\omega} &= 1 + \omega. \end{aligned}$$

We obtain

$$[\text{tr}_c](\text{Ind}_{(\Gamma, \sigma)}(\widetilde{\partial_{\mathcal{E}}^+} \otimes \nabla)) = \frac{\text{rank } \mathcal{E}}{2\pi\#(G)} \langle [\tilde{c}], [\Sigma_{g'}] \rangle.$$

When $c, [c] \in H^2(\Gamma)$, is the area 2-cocycle, then \tilde{c} is merely the restriction of the area cocycle on $PSL(2, \mathbb{R})$ to the subgroup $\Gamma_{g'}$. Then one has

$$\langle [\tilde{c}], [\Sigma_{g'}] \rangle = -2\pi\chi(\Sigma_{g'}) = 4\pi(g' - 1).$$

The corollary now follows from Theorem 5.1 above and the fact that $g' = 1 + \frac{\#(G)}{2}(2(g-1) + (n-\nu))$, $\nu = \sum_{j=1}^n 1/\nu_j$ and $\#(G) = 1 + \sum_{j=1}^n (\nu_j - 1)$. \square

We next describe the canonical pairing of $K_0(C_r^*(\Gamma, \sigma))$, given $[c] \in H^2(\Gamma)$. Since $\Sigma(g, \nu_1, \dots, \nu_n)$ is negatively curved, we know from [Ji] that

$$\mathcal{R}(\Gamma, \sigma) = \left\{ f : \Gamma \rightarrow \mathbb{C} \mid \sum_{\gamma \in \Gamma} |f(\gamma)|^2 (1 + l(\gamma))^k < \infty \text{ for all } k \geq 0 \right\},$$

where $l : \Gamma \rightarrow \mathbb{R}^+$ denotes the length function, is a dense and spectral invariant subalgebra of $C_r^*(\Gamma, \sigma)$. In particular it is closed under the smooth functional calculus, and is known as the algebra of rapidly decreasing L^2 functions on Γ . By a theorem of [Bost], the inclusion map $\mathcal{R}(\Gamma, \sigma) \subset C_r^*(\Gamma, \sigma)$ induces an isomorphism

$$K_j(\mathcal{R}(\Gamma, \sigma)) \cong K_j(C_r^*(\Gamma, \sigma)), \quad j = 0, 1.$$

As $\Sigma(g, \nu_1, \dots, \nu_n) = \underline{B}\Gamma$ is a negatively curved orbifold, we know (by [Mos] and [Gr]) that degree 2 cohomology classes in $H^2(\Gamma)$ have *bounded* representatives i.e. bounded 2-cocycles on Γ . Let c be a bounded 2-cocycle on Γ . Then it defines a cyclic 2-cocycle tr_c on the twisted group algebra $\mathbb{C}(\Gamma, \sigma)$, by a slight modification of the standard formula [CM], ([Ma] for the general case)

$$\text{tr}_c(f^0, f^1, f^2) = \sum_{g_0 g_1 g_2 = 1} f^0(g_0) f^1(g_1) f^2(g_2) c(1, g_1, g_1 g_2) \sigma(g_1, g_2).$$

Here c is assumed to be skew-symmetrized. Since the only difference with the expression obtained in [CM] is $\sigma(g_1, g_2)$, and since $|\sigma(g_1, g_2)| = 1$, we can use Lemma 6.4, part (ii) in [CM] and the assumption that c is bounded, to obtain the necessary estimates which show that in fact tr_c extends continuously to the bigger algebra $\mathcal{R}(\Gamma, \sigma)$. This induces an additive map in K -theory as before:

$$\begin{aligned} [\text{tr}_c] : K_0(\mathcal{R}(\Gamma, \sigma)) &\rightarrow \mathbb{R} \\ [\text{tr}_c]([e] - [f]) &= \widetilde{\text{tr}}_c(e, \dots, e) - \widetilde{\text{tr}}_c(f, \dots, f), \end{aligned}$$

where e, f are idempotent matrices with entries in $(\mathcal{R}(\Gamma, \sigma))^\sim$ (the unital algebra associated to $\mathcal{R}(\Gamma, \sigma)$) and $\widetilde{\text{tr}}_c$ is the canonical extension of tr_c to $(\mathcal{R}(\Gamma, \sigma))^\sim$. Observe that the twisted Kasparov map is merely

$$\mu_\sigma([\mathcal{E}]) = j_*(\text{Ind}_{(\Gamma, \sigma)}(\widetilde{\partial_\mathcal{E}^+} \otimes \nabla)) \in K_0(C^*(\Gamma, \sigma)).$$

Here $j : \mathbb{C}(\Gamma, \sigma) \otimes \mathcal{R} \rightarrow C^*(\Gamma, \sigma) \otimes \mathcal{K}$ is the natural inclusion map, and $j_* : K_0(\mathbb{C}(\Gamma, \sigma) \otimes \mathcal{R}) \rightarrow K_0(C^*(\Gamma, \sigma))$ is the induced map in K -theory. Therefore one has the equality

$$\langle [c], \mu_\sigma^{-1}[P] \rangle = \langle [\text{tr}_c], [P] \rangle$$

for any $[P] \in K_0(\mathcal{R}(\Gamma, \sigma)) \cong K_0(C_r^*(\Gamma, \sigma))$. Using the previous corollary, one has

Theorem 5.3 (Range of the higher trace on K -theory). *Let c be the area 2-cocycle on Γ . Then c is known to be a bounded 2-cocycle, and one has*

$$\langle [\text{tr}_c], [P] \rangle = \phi(\text{rank } \mathcal{E}^0 - \text{rank } \mathcal{E}^1) \in \phi\mathbb{Z},$$

where $\phi = 2(g-1) + (n-\nu) \in \mathbb{Q}$ and $\nu = \sum_{j=1}^n 1/\nu_j$. Here $[P] \in K_0(\mathcal{R}(\Gamma, \sigma)) \cong K_0(C_r^*(\Gamma, \sigma))$, and $\mathcal{E}^0, \mathcal{E}^1$ are orbifold vector bundles over $\Sigma(g, \nu_1, \dots, \nu_n)$ such that

$$\mu_\sigma^{-1}([P]) = [\mathcal{E}^0] - [\mathcal{E}^1] \in K_{orb}^0(\Sigma(g, \nu_1, \dots, \nu_n)).$$

In particular, the range of the higher trace on K -theory is

$$[\mathrm{tr}_c](K_0(C^*(\Gamma, \sigma))) = \phi \mathbb{Z}.$$

Note that ϕ is in general only a *rational* number and we will give examples to show that this is the case; however it is an *integer* whenever the orbifold is smooth, i.e. whenever $1 = \nu_1 = \dots = \nu_n$, which was the case that was considered in [CHMM]. We will apply this result in section 6 to compute the range of values the Hall conductance in the quantum Hall effect on the hyperbolic plane, for orbifold fundamental groups, extending the results in [CHMM].

5.1. Examples of orbifolds with fractional ϕ . In [Bro], Broughton has listed all the good two dimensional orbifolds which are quotients of Riemann surfaces $\Sigma_{g'}$ with genus $g' = 2$ or 3 . Using his explicit classification, we will give several examples showing that the number ϕ , as in Theorem 5.3, can be a fraction. Even when $g' = 2$, there are several examples from Table 4, [Bro], but we will focus on one of these, viz. 2.k.2, where a dihedral group of order 6 acts on the genus two Riemann surface giving rise to an orbifold of signature $(g, \nu_1, \nu_2, \nu_3, \nu_4) = (0, 2, 2, 3, 3)$. It follows that $\phi = 1/3$ in this case, and that the range of values the Hall conductance in the quantum Hall effect on the hyperbolic plane, for this particular orbifold fundamental group is $\mathbb{Z}(1/3)$. One can list all the possible denominators that can occur for ϕ by looking through the Tables 4 and 5 in [Bro], which will be used again to determine all the range of values the Hall conductance in the quantum Hall effect on the hyperbolic plane, and will be studied further in section 6.

6. THE AREA COCYCLE, THE HYPERBOLIC CONNES-KUBO FORMULA AND THE QUANTUM HALL EFFECT

In this section we prove a generalisation of the results in [CHMM] on the Quantum Hall Effect on the hyperbolic plane, where we now allow the discrete group to have torsion. We will only discuss the discrete model, as the discussion for the continuous model is similar, as shown in [CHMM]. We will first derive the discrete analogue of the hyperbolic Connes-Kubo formula for the Hall conductance 2-cocycle, which was derived in the continuous case in [CHMM]. We then relate it to the Area 2-cocycle on the twisted group algebra of the discrete Fuchsian group, and we show that these define the same cyclic cohomology class. This enables us to use the results of the previous section to show that the Hall conductance has plateaus at all energy levels belonging to any gap in the spectrum of the Hamiltonian, where it is now shown to be equal to an integral multiple of a *fractional* valued invariant. Moreover the set of possible denominators is finite and has been explicitly determined. It is plausible that this might shed light on the mathematical mechanism responsible for fractional quantum numbers.

The graph that we consider is the Cayley graph of the Fuchsian group Γ of signature (g, ν_1, \dots, ν_n) , which acts freely on the complement of a countable set of points in the hyperbolic plane. The Cayley graph embeds in the hyperbolic plane as follows. Fix a base point $u \in \mathbb{H}$ such that the stabilizer (or isotropy subgroup) at u is trivial and consider the orbit of the Γ action through u . This gives the vertices of the graph. The edges of the graph are geodesics constructed as follows. Each element of the group Γ may be written as a word of minimal length in the generators of Γ and their inverses. Each generator and its inverse determines a unique geodesic emanating from

a vertex x and these form the edges of the graph. Thus each word x in the generators determines a piecewise geodesic path from u to x . Note that if Γ has elliptic elements, then the Cayley graph of Γ has (geodesic) loops i.e. it is not a tree.

Recall that the *area cocycle* c of the Fuchsian group Γ is a canonically defined 2-cocycle on Γ that is defined as follows. Firstly, recall that there is a well known area 2-cocycle on $PSL(2, \mathbb{R})$, cf. [Co2], defined as follows: $PSL(2, \mathbb{R})$ acts on \mathbb{H} such that $\mathbb{H} \cong PSL(2, \mathbb{R})/SO(2)$. Then $c(\gamma_1, \gamma_2) = \text{Area}(\Delta(o, \gamma_1.o, \gamma_2^{-1}.o)) \in \mathbb{R}$, where o denotes an origin in \mathbb{H} and $\text{Area}(\Delta(a, b, c))$ denotes the hyperbolic area of the geodesic triangle in \mathbb{H} with vertices at $a, b, c \in \mathbb{H}$. Then the restriction of c to the subgroup Γ is the area cocycle c of Γ .

This area cocycle defines in a canonical way a cyclic 2-cocycle tr_c on the group algebra $\mathbb{C}(\Gamma, \sigma)$ as follows;

$$\text{tr}_c(a_0, a_1, a_2) = \sum_{\gamma_0 \gamma_1 \gamma_2 = 1} a_0(\gamma_0) a_1(\gamma_1) a_2(\gamma_2) c(\gamma_1, \gamma_2) \sigma(\gamma_1, \gamma_2)$$

We will now describe the hyperbolic Connes-Kubo formula for the Hall conductance in the Quantum Hall Effect. Let Ω_j denote the (diagonal) operator on $\ell^2(\Gamma)$ defined by

$$\Omega_j f(\gamma) = \Omega_j(\gamma) f(\gamma) \quad \forall f \in \ell^2(\Gamma) \quad \forall \gamma \in \Gamma$$

where

$$\Omega_j(\gamma) = \int_o^{\gamma.o} a_j \quad j = 1, \dots, 2g$$

and where $\{a_j\} \quad j = 1, \dots, 2g$ is the lift to \mathbb{H} of a symplectic basis of harmonic 1-forms on the Riemann surface of genus g underlying the orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$.

For $j = 1, \dots, 2g$, define the derivations δ_j on $\mathcal{R}(\Gamma, \sigma)$ as being the commutators $\delta_j a = [\Omega_j, a]$. A simple calculation shows that

$$\delta_j a(\gamma) = \Omega_j(\gamma) a(\gamma) \quad \forall a \in \mathcal{R}(\Gamma, \sigma) \quad \forall \gamma \in \Gamma.$$

Note that these are not inner derivations, and also that we have the simple estimate

$$|\Omega_j(\gamma)| \leq \|a_j\|_{(\infty)} d(\gamma.o, o)$$

where $d(\gamma.o, o)$ and the distance in the word metric on the group Γ , $d_\Gamma(\gamma, 1)$ are equivalent. This then yields the estimate

$$|\delta_j a(\gamma)| \leq C_N d_\Gamma(\gamma, 1)^{-N} \quad \forall N \in \mathbb{N}$$

i.e $\delta_j a \in \mathcal{R}(\Gamma, \sigma) \quad \forall a \in \mathcal{R}(\Gamma, \sigma)$. Note that since $\forall \gamma, \gamma' \in \Gamma$, the difference $\Omega_j(\gamma\gamma') - \Omega_j(\gamma')$ is a constant independent of γ' , we see that Γ -equivariance is preserved. For $j = 1, \dots, 2g$, define the cyclic 2-cocycles

$$\text{tr}_j^K(a_0, a_1, a_2) = \text{tr}(a_0(\delta_j a_1 \delta_{j+g} a_2 - \delta_{j+g} a_1 \delta_j a_2)).$$

These are supposed to give the Hall conductance for currents in the $(j+g)$ th direction which are induced by electric fields in the j th direction cf. section 6, [CHMM]. Then the *hyperbolic Connes-Kubo formula* for the Hall conductance is the cyclic 2-cocycle given by the sum

$$\text{tr}^K(a_0, a_1, a_2) = \sum_{j=1}^g \text{tr}_j^K(a_0, a_1, a_2).$$

Theorem 6.1 (The Comparison Theorem).

$$[\mathrm{tr}^K] = [\mathrm{tr}_c] \in HC^2(\mathcal{R}(\Gamma, \sigma))$$

Sketch of Proof: Our aim is now to compare the two cyclic 2-cocycles and to prove that they differ by a coboundary i.e.

$$\mathrm{tr}^K(a_0, a_1, a_2) - \mathrm{tr}_c(a_0, a_1, a_2) = b\lambda(a_0, a_1, a_2)$$

for some cyclic 1-cochain λ and where b is the cyclic coboundary operator. The key to this theorem is a geometric interpretation of the hyperbolic Connes-Kubo formula.

We begin with some calculations, to enable us to make this comparison of the cyclic 2-cocycles.

$$\begin{aligned} \mathrm{tr}^K(a_0, a_1, a_2) &= \\ \sum_{j=1}^g \sum_{\gamma_0 \gamma_1 \gamma_2 = 1} a_0(\gamma_0) (\delta_j a_1(\gamma_1) \delta_{j+g} a_2(\gamma_2) - \delta_{j+g} a_1(\gamma_1) \delta_j a_2(\gamma_2)) \sigma(\gamma_0, \gamma_1) \sigma(\gamma_0 \gamma_1, \gamma_2) \\ &= \sum_{j=1}^g \sum_{\gamma_0 \gamma_1 \gamma_2 = 1} a_0(\gamma_0) a_1(\gamma_1) a_2(\gamma_2) (\Omega_j(\gamma_1) \Omega_{j+g}(\gamma_2) - \Omega_{j+g}(\gamma_1) \Omega_j(\gamma_2)) \sigma(\gamma_1, \gamma_2) \end{aligned}$$

since by the cocycle identity for multipliers, one has

$$\begin{aligned} \sigma(\gamma_0, \gamma_1) \sigma(\gamma_0 \gamma_1, \gamma_2) &= \sigma(\gamma_0, \gamma_1 \gamma_2) \sigma(\gamma_1, \gamma_2) \\ &= \sigma(\gamma_0, \gamma_0^{-1}) \sigma(\gamma_1, \gamma_2) \quad \text{since } \gamma_0 \gamma_1 \gamma_2 = 1 \\ &= \sigma(\gamma_1, \gamma_2) \quad \text{since } \sigma(\gamma_0, \gamma_0^{-1}) = 1. \end{aligned}$$

So we are now in a position to compare the two cyclic 2-cocycles. Define $\Psi_j(\gamma_1, \gamma_2) = \Omega_j(\gamma_1) \Omega_{j+g}(\gamma_2) - \Omega_{j+g}(\gamma_1) \Omega_j(\gamma_2)$. Let $\Xi : \mathbb{H} \rightarrow \mathbb{R}^{2g}$ denote the lift of the Abel-Jacobi map. It is a symplectic map, since it is known to be a holomorphic embedding. Therefore if ω and ω_J are their respective symplectic 2-forms, then one has $\Xi^*(\omega_J) = \omega$. Then one has the following geometric lemma

Lemma 6.2.

$$\sum_{j=1}^g \Psi_j(\gamma_1, \gamma_2) = \int_{\Delta_E(\gamma_1, \gamma_2)} \omega_J$$

where $\Delta_E(\gamma_1, \gamma_2)$ denotes the Euclidean triangle with vertices at $\Xi(o), \Xi(\gamma_1.o)$ and $\Xi(\gamma_2.o)$, and ω_J denotes the flat Kähler 2-form on the Jacobi variety. That is, $\sum_{j=1}^g \Psi_j(\gamma_1, \gamma_2)$ is equal to the Euclidean area of the Euclidean triangle $\Delta_E(\gamma_1, \gamma_2)$.

Proof. We need to consider the expression

$$\sum_{j=1}^g \Psi_j(\gamma_1, \gamma_2) = \sum_{j=1}^g \Omega_j(\gamma_1) \Omega_{j+g}(\gamma_2) - \Omega_{j+g}(\gamma_1) \Omega_j(\gamma_2).$$

Let s denote the symplectic form on \mathbb{R}^{2g} given by:

$$s(u, v) = \sum_{j=1}^g (u_j v_{j+g} - u_{j+g} v_j).$$

The so-called ‘symplectic area’ of a triangle with vertices $\Xi(o) = 0, \Xi(\gamma_1.o), \Xi(\gamma_2.o)$ may be seen to be $s(\Xi(\gamma_1.o), \Xi(\gamma_2.o))$. To appreciate this, however, we need to utilise an argument from [GH], pages 333-336. In terms of the standard basis of \mathbb{R}^{2g} (given in this case by vertices in the integer period lattice arising from our choice of basis of harmonic one forms) and corresponding coordinates u_1, u_2, \dots, u_{2g} the form s is the two form on \mathbb{R}^{2g} given by

$$\omega_J = \sum_{j=1}^g du_j \wedge du_{j+g}.$$

Now the ‘symplectic area’ of a triangle in \mathbb{R}^{2g} with vertices $\Xi(o) = 0, \Xi(\gamma_1.o), \Xi(\gamma_2.o)$ is given by integrating ω_J over the triangle and a brief calculation reveals that this yields $s(\Xi(\gamma_1.o), \Xi(\gamma_2.o))/2$, proving the lemma. \square

We also observe that since $\omega = \Xi^* \omega_J$, one has

$$c(\gamma_1, \gamma_2) = \int_{\Delta(\gamma_1, \gamma_2)} \omega = \int_{\Xi(\Delta(\gamma_1, \gamma_2))} \omega_J$$

Therefore the difference

$$\begin{aligned} \sum_{j=1}^g \Psi_j(\gamma_1, \gamma_2) - c(\gamma_1, \gamma_2) &= \int_{\Delta_E(\gamma_1, \gamma_2)} \omega_J - \int_{\Xi(\Delta(\gamma_1, \gamma_2))} \omega_J \\ &= \int_{\partial \Delta_E(\gamma_1, \gamma_2)} \Theta_J - \int_{\partial \Xi(\Delta(\gamma_1, \gamma_2))} \Theta_J \end{aligned}$$

where Θ_J is a 1-form on the universal cover of the Jacobi variety such that $d\Theta_J = \omega_J$. Therefore one has

$$\sum_{j=1}^g \Psi_j(\gamma_1, \gamma_2) - c(\gamma_1, \gamma_2) = h(1, \gamma_1) - h(\gamma_1^{-1}, \gamma_2) + h(\gamma_2^{-1}, 1)$$

where $h(\gamma_1^{-1}, \gamma_2) = \int_{\Xi(\ell(\gamma_1, \gamma_2))} \Theta_J - \int_{m(\gamma_1, \gamma_2)} \Theta_J$, where $\ell(\gamma_1, \gamma_2)$ denotes the unique geodesic in \mathbb{H} joining $\gamma_1.o$ and $\gamma_2.o$ and $m(\gamma_1, \gamma_2)$ is the straight line in the Jacobi variety joining the points $\Xi(\gamma_1.o)$ and $\Xi(\gamma_2.o)$. Since we can also write $h(\gamma_1^{-1}, \gamma_2) = \int_{D(\gamma_1, \gamma_2)} \omega_J$, where $D(\gamma_1, \gamma_2)$ is a disk in the Jacobi variety with boundary $\Xi(\ell(\gamma_1, \gamma_2)) \cup m(\gamma_1, \gamma_2)$, we see that h is Γ -invariant.

We now define the cyclic 1-cochain λ on $\mathcal{R}(\Gamma, \sigma)$ as

$$\lambda(a_0, a_1) = \text{tr}((a_0)_h a_1) = \sum_{\gamma_0 \gamma_1 = 1} h(1, \gamma_1) a_0(\gamma_0) a_1(\gamma_1) \sigma(\gamma_0, \sigma_1)$$

where $(a_0)_h$ is the operator on $\ell^2(\Gamma)$ whose matrix in the canonical basis is $h(\gamma_1, \gamma_2) a_0(\gamma_1 \gamma_2^{-1})$. Firstly, one has by definition

$$b\lambda(a_0, a_1, a_2) = \lambda(a_0 a_1, a_2) - \lambda(a_0, a_1 a_2) + \lambda(a_2 a_0, a_1)$$

We compute each of the terms separately

$$\begin{aligned}\lambda(a_0 a_1, a_2) &= \sum_{\gamma_0 \gamma_1 \gamma_2 = 1} h(1, \gamma_2) a_0(\gamma_0) a_1(\gamma_1) a_2(\gamma_2) \sigma(\gamma_1, \gamma_2) \\ \lambda(a_0, a_1 a_2) &= \sum_{\gamma_0 \gamma_1 \gamma_2 = 1} h(1, \gamma_1 \gamma_2) a_0(\gamma_0) a_1(\gamma_1) a_2(\gamma_2) \sigma(\gamma_1, \gamma_2) \\ \lambda(a_2 a_0, a_1) &= \sum_{\gamma_0 \gamma_1 \gamma_2 = 1} h(1, \gamma_1) a_0(\gamma_0) a_1(\gamma_1) a_2(\gamma_2) \sigma(\gamma_1, \gamma_2)\end{aligned}$$

Now by Γ -equivariance, $h(1, \gamma_1 \gamma_2) = h(\gamma_1^{-1}, \gamma_2)$ and $h(1, \gamma_2) = h(\gamma_2^{-1}, 1)$. Therefore one has

$$\begin{aligned}b\lambda(a_0, a_1, a_2) &= \\ \sum_{\gamma_0 \gamma_1 \gamma_2 = 1} a_0(\gamma_0) a_1(\gamma_1) a_2(\gamma_2) &\left(h(\gamma_2^{-1}, 1) - h(\gamma_1^{-1}, \gamma_2) + h(1, \gamma_1) \right) \sigma(\gamma_1, \gamma_2)\end{aligned}$$

Using the formula above, we see that

$$b\lambda(a_0, a_1, a_2) = \text{tr}^K(a_0, a_1, a_2) - \text{tr}_c(a_0, a_1, a_2).$$

It follows from Connes pairing theory of cyclic cohomology and K -theory [Co2] and the Comparison Theorem above that

Corollary 6.3.

$$\text{tr}^K(P, P, P) = \text{tr}_c(P, P, P)$$

for all projections $P \in \mathcal{R}(\Gamma, \sigma)$.

Recall that by the range of the higher trace Theorem 5.3, one has

$$\text{tr}_c(P, P, P) \in \phi \mathbb{Z} \tag{6.1}$$

for all projections $P \in \mathcal{R}(\Gamma, \sigma)$, where $\phi = 2(g - 1) + (n - \nu) \in \mathbb{Q}$.

Finally, suppose that we are given a very thin hyperboloid sample of pure metal, with electrons situated along the Cayley graph of Γ , and a very strong magnetic field which is uniform and normal in direction to the sample. Then at very low temperatures, close to absolute zero, quantum mechanics dominates and the discrete model that is considered here is a model of electrons moving on the Cayley graph of Γ which is embedded in the hyperboloid. The associated discrete Hamiltonian H_σ for the electron in the magnetic field is given by the Random Walk operator in the projective (Γ, σ) regular representation on the Cayley graph of the group Γ . It is also known as the generalized *Harper operator* and was first studied in this generalized context in [Sun] and also in [CHMM]. We will see that the Hamiltonian that we consider is in a natural way the sum of a free Hamiltonian and an interacting term.

The hyperbolic Connes-Kubo formula for the Hall conductance σ_E at the energy level E is defined as follows; let $P_E = \chi_{[0, E]}(H_\sigma)$ be the spectral projections of the Hamiltonian to energy levels less than or equal to E . Then if $E \notin \text{spec}(H_\sigma)$, one can show that $P_E \in \mathcal{R}(\Gamma, \sigma)$, and the Hall conductance is defined as

$$\sigma_E = \text{tr}^K(P_E, P_E, P_E).$$

As mentioned earlier, it measures the sum of the contributions to the Hall conductance at the energy level E for currents in the $(j+g)$ th direction which are induced by electric fields in the j th direction, cf. section 6 [CHMM]. When this is combined with equation (6.1), one sees that the Hall conductance takes on values in $\phi\mathbb{Z}$ whenever the energy level E lies in a gap in the spectrum of the Hamiltonian H_σ . In fact we notice that the Hall conductance is a constant function of the energy level E for all values of E in the same gap in the spectrum of the Hamiltonian. That is, the Hall conductance has plateaus which are *integer* multiples of the fraction ϕ on the gap in the spectrum of the Hamiltonian.

We now give some details. Recall the left σ -regular representation

$$(\lambda(\gamma)f)(\gamma') = f(\gamma^{-1}\gamma')\sigma(\gamma', \gamma^{-1}\gamma')$$

$\forall f \in \ell^2(\Gamma)$ and $\forall \gamma, \gamma' \in \Gamma$. It has the property that

$$\lambda(\gamma)\lambda(\gamma') = \sigma(\gamma, \gamma')\lambda(\gamma\gamma')$$

Let $S = \{A_j, B_j, A_j^{-1}, B_j^{-1}, C_i, C_i^{-1} : j = 1, \dots, g, \quad i = 1, \dots, n\}$ be a symmetric set of generators for Γ . Then the Hamiltonian is explicitly given as

$$H_\sigma : \ell^2(\Gamma) \rightarrow \ell^2(\Gamma)$$

$$H_\sigma = \sum_{\gamma \in S} \lambda(\gamma)$$

and is clearly by definition a bounded self adjoint operator. Notice that the Hamiltonian can be decomposed as a sum of a free Hamiltonian containing the torsionfree generators and an interaction term containing the torsion generators.

$$H_\sigma = H_\sigma^{free} + H_\sigma^{interaction}$$

where

$$H_\sigma^{free} = \sum_{\gamma \in S'} \lambda(\gamma) \quad \text{and} \quad H_\sigma^{interaction} = \sum_{\gamma \in S''} \lambda(\gamma)$$

and where $S' = \{A_j, B_j, A_j^{-1}, B_j^{-1} : j = 1, \dots, g\}$ and $S'' = \{C_i, C_i^{-1} : i = 1, \dots, n\}$.

Lemma 6.4. *If $E \notin \text{spec}(H_\sigma)$, then $P_E \in \mathcal{R}(\Gamma, \sigma)$, where $P_E = \chi_{[0, E]}(H_\sigma)$ is the spectral projection of the Hamiltonian to energy levels less than or equal to E .*

Proof. Since $E \notin \text{spec}(H_\sigma)$, then $P_E = \chi_{[0, E]}(H_\sigma) = \phi(H_\sigma)$ for some smooth, compactly supported function ϕ . Now by definition, $H_\sigma \in \mathbb{C}(\Gamma, \sigma) \subset \mathcal{R}(\Gamma, \sigma)$, and since $\mathcal{R}(\Gamma, \sigma)$ is closed under the smooth functional calculus by the result of [Ji], it follows that $P_E \in \mathcal{R}(\Gamma, \sigma)$. □

Therefore by the range of the higher trace Theorem 5.3, and the discussion above, we see that

Theorem 6.5 (Generalized Quantum Hall Effect). *Suppose that the energy level E lies in a gap of the spectrum of the Hamiltonian H_σ , then the Hall conductance*

$$\sigma_E = \text{tr}^K(P_E, P_E, P_E) = \text{tr}_c(P_E, P_E, P_E) \in \phi\mathbb{Z}$$

That is, the Hall conductance has plateaus which are integer multiples of ϕ on any gap in the spectrum of the Hamiltonian, where $\phi = 2(g - 1) + (n - \nu) \in \mathbb{Q}$.

Remarks 6.6. The set of possible denominators for ϕ is finite and has been explicitly determined in [Bro]. It is plausible that this Theorem might shed light on the mathematical mechanism responsible for fractional quantum numbers that occur in the Quantum Hall Effect.

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